Worked Examples of Laplace Transform and Convolution

Problem 1: Solve the differential equation: $\ddot{x}+3\dot{x}+2x=2e^{-t}, \quad x(0)=0, \quad \dot{x}(0)=0$

Plan: This problem is certainly most easily solved using other methods, but it should help to illustrate how the Laplace transform and convolution are applied to the solution of an ordinary differential equation.

The methodology is based on <u>two basic ideas</u>.

Idea #1: Find a way to transform a differential equation into an algebraic equation. Then solve it using basic algebra. Then transform back to get the desired solution. We do this via the ***Laplace transform***.

Idea #2: Solve the given system for its unit impulse response, then find a way to use this to solve the system for any given input signal. We do this via ***convolution***.

The two essential definitions are these:

Convolution: Given two functions $w(t)$ and $f(t)$, define $\boxed{(w*f)(t)=\int_0^t w(t-\tau)f(\tau)d\tau}$. We showed in class that if we can solve the differential equation $p(D)x=\delta(t)$ for the unit impulse response $w(t)$ (also called the ***weight function***), then the differential equation $p(D)x=f(t)$ will have the solution $(w*f)(t)$. This fact is also known as ***Green's Formula***.

Laplace Transform: Given a function $f(t)$, we define its Laplace transform to be the function $\boxed{F(s)=[\mathcal{L}(f)](s)=\int_0^\infty e^{-st}f(t)dt}$. This function is generally only defined for come subset of values of the (complex) parameter s known as its **region of convergence**.

The usefulness of this transform method is built on the fact that we can relatively easily find the Laplace transform for most everything that appears in a given differential equation of the form $p(D)x=f(t)$, and once we have a table of these transforms we can generally invert the process by inspection. Another essential fact is that the Laplace transform acts linearly, and this allows us to decompose complex problems into a sums of simple problems.

The Old (and Very Good) Solution: For $\ddot{x}+3\dot{x}+2x=2e^{-t}, \quad x(0)=0, \quad \dot{x}(0)=0$, the homogeneous equation $\ddot{x}+3\dot{x}+2x=0$ is easy to solve. Its characteristic polynomial is $p(s)=s^2+3s+2=(s+2)(s+1)$ which yields the two roots $s=-2$ and $s=-1$. This gives the two independent solutions e^{-2t} and e^{-t}, and all homogeneous solutions are of the form $x_h(t)=c_1e^{-2t}+c_2e^{-t}$. Note that both of these homogeneous solutions are <u>transient</u> in the sense that they decay exponentially as t increases.

Next, we need to find a particular solution $x_p(t)$ that satisfies the inhomogeneous differential equation. One look at the right-hand-side and we see that the Exponential Response Formula (ERF) won't work – there is resonance. We can, however, use the Resonant Response Formula to get the particular solution $x_p(t)=\dfrac{2te^{-t}}{p'(-1)}=\dfrac{2te^{-t}}{1}=2te^{-t}$, so the general solution is $x(t)=x_h(t)+x_p(t)=c_1e^{-2t}+c_2e^{-t}+2te^{-t}$. Its derivative is $\dot{x}(t)=-2c_1e^{-2t}-c_2e^{-t}-2te^{-t}+2e^{-t}$. Substituting the (rest) initial conditions gives $\left\{\begin{array}{l} x(0)=c_1+c_2=0 \\ \dot{x}(0)=-2c_1-c_2+2=0 \end{array}\right\}$, and these can be solved to give $c_1=2, c_2=-2$, so the solution is $\boxed{x(t)=2e^{-2t}-2e^{-t}+2te^{-t}}$.

Solving directly by Laplace transform: We calculated the following Laplace transforms:

(1) $\mathcal{L}(e^{kt}) = \dfrac{1}{s-k}$ with region of convergence $\text{Re}(s) > k$, so $\mathcal{L}(e^{-2t}) = \dfrac{1}{s+2}$.

(2) If the Laplace transform of $x(t)$ is $X(s)$, then the Laplace transforms of its derivatives are $\mathcal{L}(\dot{x}(t)) = sX(s) - x(0-)$ and $\mathcal{L}(\ddot{x}(t)) = s^2X(s) - s\,x(0-) - \dot{x}(0-)$. In the case of rest initial conditions $x(0-) = \dot{x}(0-) = 0$, these are greatly simplified and, in fact $\mathcal{L}(p(D)x) = p(s)X(s)$. Specifically, $\mathcal{L}(\ddot{x} + 3\dot{x} + 2x) = s^2X(s) + 3sX(s) + 2X(s) = (s^2 + 3s + 2)X(s) = p(s)X(s)$.

If we now transform the entire differential equation, we get $(s^2 + 3s + 2)X(s) = \dfrac{2}{s+1}$.

We then solve for $X(s) = \dfrac{2}{(s+1)(s^2+3s+2)} = \dfrac{2}{(s+2)(s+1)^2} = \dfrac{A}{s+2} + \dfrac{B}{s+1} + \dfrac{C}{(s+1)^2}$.

There are many good ways to find the unknowns *A*, *B*, and *C*. For example, if we multiply through by the common denominator to clear fractions, we get $2 = A(s+1)^2 + B(s+1)(s+2) + C(s+2)$. Plugging in the specific values $s = -2$ and $s = -1$ quickly yields that $A = 2$ and $C = 2$. Plugging in, for example, $s = 0$ and using the values for *A* and *C* then yields $B = -2$. So $X(s) = \dfrac{2}{s+2} - \dfrac{2}{s+1} + \dfrac{2}{(s+1)^2}$.

Consulting our table of common Laplace transforms, we see that $\dfrac{2}{s+2} = \mathcal{L}(2e^{-2t})$, $\dfrac{2}{s+1} = \mathcal{L}(2e^{-t})$, and $\dfrac{2}{(s+1)^2} = \mathcal{L}(2te^{-t})$, so transforming back gives $\boxed{x(t) = 2e^{-2t} - 2e^{-t} + 2te^{-t}}$.

Solution using unit impulse response and convolution: In class we solved the equation $\ddot{x} + 3\dot{x} + 2x = \delta(t)$ with rest initial conditions $x(0) = 0, \dot{x}(0) = 0$ to find the weight function $w(t) = \begin{Bmatrix} 0 & t < 0 \\ -e^{-2t} + e^{-t} & t > 0 \end{Bmatrix}$. This is also called the unit impulse response. Note that the Laplace transform of $w(t)$ is $W(s) = \dfrac{1}{p(s)}$ where $p(s)$ is the characteristic polynomial for this system.

If we use $f(t) = 2e^{-t}$, then convolution of the weight function and the given input signal gives:

$$(w * f)(t) = \int_{\tau=0}^{\tau=t} w(t-\tau)f(\tau)d\tau = \int_{\tau=0}^{\tau=t} 2e^{-\tau}(-e^{-2(t-\tau)} + e^{-(t-\tau)})d\tau$$
$$= \int_{\tau=0}^{\tau=t} (-2e^{-2t}e^{\tau} + 2e^{-t})d\tau = -2e^{-2t}(e^{t} - 1) + 2e^{-2t}(t-0) = \boxed{-2e^{-t} + 2e^{-2t} + 2te^{-2t} = x(t)}$$

Problem 2: Solve the differential equation: $\ddot{x} + \dot{x} - 2x = 3e^{-t}\cos 2t, \quad x(0) = 0, \quad \dot{x}(0) = 0$

The characteristic polynomial in this case is $p(s) = s^2 + s - 2 = (s+2)(s-1)$. If we solve this problem using the Laplace transform with $W(s)$ as the Laplace transform of $x(t)$, then with rest initial conditions the left-hand-side will have transform $p(s)X(s) = (s+2)(s-1)X(s)$. We know that $\mathcal{L}(\cos\omega t) = \dfrac{s}{s^2 + \omega^2}$, so using the exponential shift rule and linearity, $\mathcal{L}(3e^{-t}\cos 2t) = \dfrac{3(s+1)}{(s+1)^2 + 4}$. The entire differential equation therefore transforms to $(s+2)(s-1)X(s)=\dfrac{3(s+1)}{(s+1)^2+4}$ or $X(s)=\dfrac{3(s+1)}{[(s+1)^2+4](s+2)(s-1)}$, a rational function. In the complex plane, this function has poles at $s=1$, $s=-2$, $s=-1+2i$, and $s=-1-2i$.

It's worth noting, even before developing the explicit solution, that qualitatively the fact that there's one pole with positive real part ($s=1$) means that the solution will be unstable in the long term (exponential growth). The other poles indicate transient behavior, exponential decay and decaying oscillations (for the complex conjugate pair of poles).

The next step is to carry out the partial fractions decomposition. It is helpful to remain ever mindful of the ultimate goal of having the terms be easily recognizable as Laplace transforms of familiar functions so that we can invert the process and find the solution. With this in mind, let's express the partial fractions decomposition in the following form:

$$X(s)=\frac{3s+3}{[(s+1)^2+4](s+2)(s-1)}=\frac{A(s+1)+B}{(s+1)^2+4}+\frac{C}{s+2}+\frac{D}{s-1}$$

The reason for this choice is that we know that $\mathcal{L}(\cos 2t)=\dfrac{s}{s^2+4}$ and $\mathcal{L}(e^{-t}\cos 2t)=\dfrac{s+1}{(s+1)^2+4}$; and $\mathcal{L}(\sin 2t)=\dfrac{2}{s^2+4}$ and $\mathcal{L}(e^{-t}\sin 2t)=\dfrac{2}{(s+1)^2+4}$; and $\mathcal{L}(e^{-2t})=\dfrac{1}{s+2}$ and $\mathcal{L}(e^{t})=\dfrac{1}{s-1}$.

Let's clear the fractions by multiplying by the common denominator. This yields:

$$3s+3=[A(s+1)+B](s+2)(s-1)+C[(s+1)^2+4](s-1)+D[(s+1)^2+4](s+2)$$

Depending on your algebraic tastes, there are many ways to proceed to find the unknowns. You could, for example, multiply everything out, collect terms, match coefficients, and then solve a system of four linear equations in four unknowns to find *A*, *B*, *C*, and *D*. Perhaps a simpler approach is to substitute specific values for *s* that will yield simple relations. [This is the basis of the Heaviside "cover-up" method.]

$s=1$ yields $6=24D \Rightarrow \boxed{D=\tfrac{1}{4}}$ and $s=-2$ yields $-3=-15C \Rightarrow \boxed{C=\tfrac{1}{5}}$

After these, the choices are less obvious, but any choices will yield usable equations. For example,

$s=-1$ yields $0=-2B-8C+4D=-2B-\tfrac{8}{5}+1=-2B-\tfrac{3}{5} \Rightarrow 2B=-\tfrac{3}{5} \Rightarrow \boxed{B=-\tfrac{3}{10}}$

$s=0$ yields $3=-2(A+B)-5C+10D=-2A-2B-5C+10D=-2A+\tfrac{3}{5}-1+\tfrac{5}{2} \Rightarrow 2A=-\tfrac{9}{10} \Rightarrow \boxed{A=-\tfrac{9}{20}}$

So we have $X(s)=\dfrac{-\frac{9}{20}(s+1)-\frac{3}{10}}{(s+1)^2+4}+\dfrac{\frac{1}{5}}{s+2}+\dfrac{\frac{1}{4}}{s-1}=-\frac{9}{20}\left(\dfrac{s+1}{(s+1)^2+4}\right)-\frac{3}{20}\left(\dfrac{2}{(s+1)^2+4}\right)+\frac{1}{5}\left(\dfrac{1}{s+2}\right)+\frac{1}{4}\left(\dfrac{1}{s-1}\right)$.

Note, in particular, the adjustment in the 2nd term to conform to the fact that $\mathcal{L}(e^{-t}\sin 2t)=\dfrac{2}{(s+1)^2+4}$.

We can now use linearity and the fact that each of these terms are multiples of known transforms to conclude that the solution is $\boxed{x(t)=-\tfrac{9}{20}(e^{-t}\cos 2t)-\tfrac{3}{20}(e^{-t}\sin 2t)+\tfrac{1}{5}e^{-2t}+\tfrac{1}{4}e^{t}}$.

It's worth noting that the examples presented here could have been done without transform methods or convolution. The real utility of these methods is in dealing with more irregular input signals and in understanding qualitatively the long-term behavior of solutions by analyzing the poles.