Laplace Transform

Properties of the Laplace transform

0. Definition: $\mathcal{L}[f(t)] = F(s) = \int_{0-}^{\infty} e^{-st} f(t)dt$ for $\operatorname{Re}(s) \gg 0$.

1. Linearity: $\mathcal{L}[af(t)+bg(t)] = a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)] = aF(s)+bG(s)$.

2. Inverse transform: $F(s)$ essentially determines $f(t)$.

3. s-shift rule: $\mathcal{L}[e^{rt} f(t)] = F(s-r)$.

4. t-shift rule: $\mathcal{L}[f(t-a)] = e^{-as}F(s)$ if $a \ge 0$ and $f(t) = 0$ for $t < 0$.
   This may also be expressed as $\mathcal{L}[f_a(t)] = e^{-as}F(s)$ where $f_a(t) = u(t-a)f(t-a) = \begin{Bmatrix} f(t-a) & \text{if } t > a \\ 0 & \text{if } t < a \end{Bmatrix}$.

5. s-derivative rule: $\mathcal{L}[t f(t)] = -F'(s)$.

6. t-derivative rule: $\mathcal{L}[f'(t)] = sF(s) - f(0-)$
$$\mathcal{L}[f''(t)] = s^2F(s) - sf(0-) - f'(0-)$$
$$\mathcal{L}[f^{(n)}(t)] = s^nF(s) - s^{n-1}f(0-) - s^{n-2}f'(0-) - \cdots - f^{(n-1)}(0-)$$

7. Convolution rules: $\mathcal{L}[f(t)*g(t)] = F(s)G(s)$, $(f*g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau = \int_0^t f(\tau)g(t-\tau)d\tau$.

8. Weight function: $\mathcal{L}[w(t)] = W(s)$, $w(t)$ the unit impulse response.
   If $q(t)$ is regarded as the input signal in $p(D)x = q(t)$, $W(s) = \frac{1}{p(s)}$.

Formulas for the Laplace transform

$$\mathcal{L}[1] = \frac{1}{s}$$

$$\mathcal{L}[\delta(t)] = 1$$

$$\mathcal{L}[\delta(t-a)] = \mathcal{L}[\delta_a(t)] = e^{-as}$$

$$\mathcal{L}[u(t-a)] = \mathcal{L}[u_a(t)] = \frac{e^{-as}}{s}$$

$$\mathcal{L}[e^{at}] = \frac{1}{s-a}$$

$$\mathcal{L}[t] = \frac{1}{s^2}$$

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}[t^n f(t)] = (-1)^n F^{(n)}(s)$$

$$\mathcal{L}[u(t-a)f(t-a)] = e^{-as}F(s)$$

$$\mathcal{L}[u(t-a)f(t)] = e^{-as}\mathcal{L}[f(t+a)]$$

$$\mathcal{L}[\cos(\omega t)] = \frac{s}{s^2+\omega^2}$$

$$\mathcal{L}[\sin(\omega t)] = \frac{\omega}{s^2+\omega^2}$$

$$\mathcal{L}[t\cos(\omega t)] = \frac{s^2-\omega^2}{(s^2+\omega^2)^2}$$

$$\mathcal{L}[t\sin(\omega t)] = \frac{2\omega s}{(s^2+\omega^2)^2}$$

$$\mathcal{L}[e^{zt}\cos(\omega t)] = \frac{s-z}{(s-z)^2+\omega^2}$$

$$\mathcal{L}[e^{zt}\sin(\omega t)] = \frac{\omega}{(s-z)^2+\omega^2}$$

where $u(t)$ is the unit step function $u(t) = 1$ for $t > 0$, $u(t) = 0$ for $t < 0$.