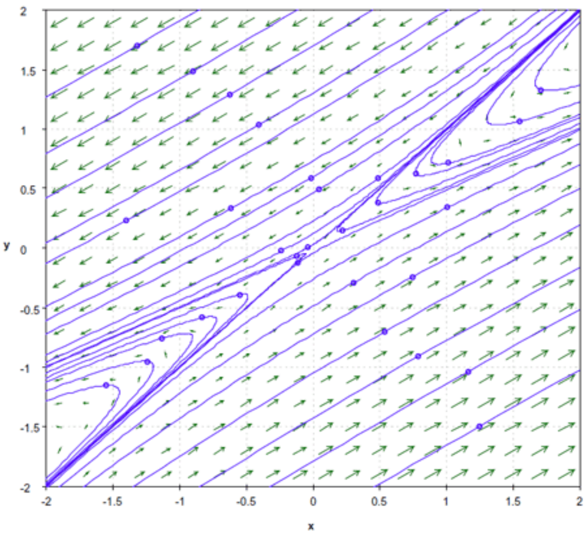
Matrix Methods for Solving Systems of 1st Order Linear Differential Equations

The Main Idea:

Given a system of 1st order linear differential equations $\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$ with initial conditions $\mathbf{x}(0)$, we use eigenvalue-eigenvector analysis to find an appropriate basis $\mathcal{B} = \{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ for $\mathbf{R}^n$ and a change of basis matrix $\mathbf{S} = \begin{bmatrix} \uparrow & & \uparrow \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ \downarrow & & \downarrow \end{bmatrix}$ such that in coordinates relative to this basis ($\mathbf{u} = \mathbf{S}^{-1}\mathbf{x}$) the system is in a standard form with a known solution. Specifically, we find a standard matrix $\mathbf{B} = [\mathbf{A}]_{\mathcal{B}} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$, transform the system into $\frac{d\mathbf{u}}{dt} = \mathbf{B}\mathbf{u}$, solve it as $\mathbf{u}(t) = [e^{t\mathbf{B}}]\mathbf{u}(0)$ where $[e^{t\mathbf{B}}]$ is the ***evolution matrix*** for **B**, then transform back to the original coordinates to get $\mathbf{x}(t) = [e^{t\mathbf{A}}]\mathbf{x}(0)$ where $[e^{t\mathbf{A}}] = \mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1}$ is the ***evolution matrix*** for **B**. That is $\mathbf{x}(t) = [e^{t\mathbf{A}}] = \mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1}\mathbf{x}(0)$. This is actually easier to do than it is to explain, so here are a few illustrative examples:

The diagonalizable case



Problem: Solve the system $\left\{ \begin{array}{l} \frac{dx}{dt} = 5x - 6y \\ \frac{dy}{dt} = 3x - 4y \end{array} \right\}$ with initial conditions $x(0) = 3,\ y(0) = 1$.

Solution: In matrix form, we have $\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$ where $\mathbf{A} = \begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix}$ and $\mathbf{x}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. We start by finding the eigenvalues of the matrix: $\lambda\mathbf{I} - \mathbf{A} = \begin{bmatrix} \lambda - 5 & 6 \\ -3 & \lambda + 4 \end{bmatrix}$, and the characteristic polynomial is $p_{\mathbf{A}}(\lambda) = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1)$. This gives the eigenvalues $\lambda_1 = 2$ and $\lambda_2 = -1$. The first of these gives the eigenvector $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, and the second gives the eigenvector $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. So we have $\left\{ \begin{array}{l} \mathbf{A}\mathbf{v}_1 = \lambda_1\mathbf{v}_1 \\ \mathbf{A}\mathbf{v}_2 = \lambda_2\mathbf{v}_2 \end{array} \right\}$. The change of basis matrix is $\mathbf{S} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and with the new basis (of eigenvectors) $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ we have $[\mathbf{A}]_{\mathcal{B}} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} = \mathbf{D}$, a diagonal matrix. [There is no need to carry out the multiplication of the matrices if $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ is known to be is a basis of eigenvectors. It will always yield a diagonal matrix with the eigenvalues on the diagonal.]

The evolution matrix for this diagonal matrix is $[e^{t\mathbf{D}}] = \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{bmatrix}$, and the solution of the system is:

$$\mathbf{x}(t) = [e^{t\mathbf{A}}]\mathbf{x}(0) = \mathbf{S}[e^{t\mathbf{D}}]\mathbf{S}^{-1}\mathbf{x}(0) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2e^{2t} & e^{-t} \\ e^{2t} & e^{-t} \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 4e^{2t} - e^{-t} \\ 2e^{2t} - e^{-t} \end{bmatrix} = 2e^{2t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} - e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 2e^{2t}\mathbf{v}_1 - e^{-t}\mathbf{v}_2$$

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The complex eigenvalue case

Let **A** be a matrix with a complex conjugate pair of eigenvalues $\lambda = a + ib$ and $\bar{\lambda} = a - ib$. We can proceed just as in the case of real eigenvalues and find a complex vector **w** such that $(\lambda\mathbf{I} - \mathbf{A})\mathbf{w} = \mathbf{0}$. The components of such a vector **w** will have complex numbers for its components. If we decompose **w** into its real and imaginary vector components as $\mathbf{w} = \mathbf{u} + i\mathbf{v}$ (where **u** and **v** and real vectors), we can calculate that:

(1) $$\mathbf{Aw} = \mathbf{Au} + i\mathbf{Av} = \lambda\mathbf{w} = (a+ib)(\mathbf{u}+i\mathbf{v}) = (a\mathbf{u} - b\mathbf{v}) + i(b\mathbf{u} + a\mathbf{v})$$

If we define the vector $\hat{\mathbf{w}} = \mathbf{u} - i\mathbf{v}$ and use the easy-to-prove fact that for a matrix **A** with all real entries we'll have $\overline{\mathbf{Aw}} = \mathbf{A}\hat{\mathbf{w}} = \overline{\lambda\mathbf{w}} = \bar{\lambda}\hat{\mathbf{w}}$, we see that $\hat{\mathbf{w}} = \mathbf{u} - i\mathbf{v}$ will also be an eigenvector with eigenvalue $\bar{\lambda}$, and:

(2) $$\mathbf{Au} - i\mathbf{Av} = (a\mathbf{u} - b\mathbf{v}) - i(b\mathbf{u} + a\mathbf{v})$$

The true value of this excursion into the world of complex numbers and complex vectors is seen when we add and subtract equation (1) and (2). We get:

$$2\mathbf{Au} = 2(a\mathbf{u} - b\mathbf{v})$$
$$2i\mathbf{Av} = 2i(b\mathbf{u} + a\mathbf{v})$$

After cancellation of the factors of 2 and 2*i* in the respective equations and rearranging, we get:

$$\mathbf{Av} = a\mathbf{v} + b\mathbf{u}$$
$$\mathbf{Au} = -b\mathbf{v} + a\mathbf{u}$$

Note that we are now back in the "real world": all vectors and scalars in the above equations are real. If we use the two vectors $\mathcal{B} = \{\mathbf{v}, \mathbf{u}\}$ as basis vectors associated with the two complex conjugate eigenvalues, we see that in coordinates associated with this basis (and change of basis matrix $\mathbf{S} = [\mathbf{v} \;\; \mathbf{u}]$) we'll have the matrix $[\mathbf{A}]_{\mathcal{B}}$ of the form:

$$[\mathbf{A}]_{\mathcal{B}} = \mathbf{S}^{-1}\mathbf{AS} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \sqrt{a^2+b^2}\begin{bmatrix} a/\sqrt{a^2+b^2} & -b/\sqrt{a^2+b^2} \\ b/\sqrt{a^2+b^2} & a/\sqrt{a^2+b^2} \end{bmatrix} = |\lambda|\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = |\lambda|\mathbf{R}_\theta$$

where $\mathbf{R}_\theta$ is the rotation matrix corresponding to the angle $\theta = \arg(\lambda)$.

<u>Next, we want to find the evolution matrix for this (real) normal form</u>.

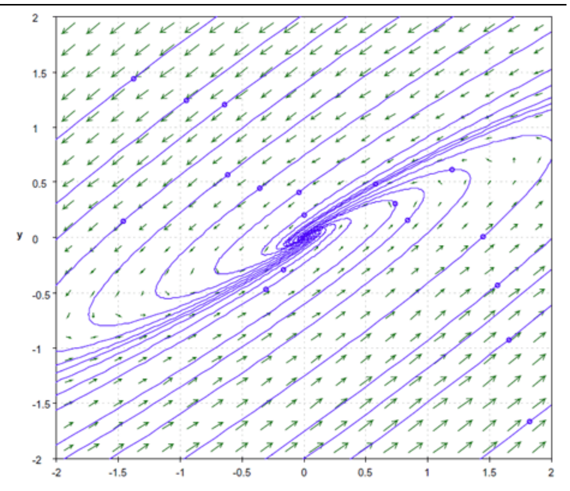
In fact, $[e^{t\mathbf{B}}] = e^{at}\begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix}$, a time-varying rotation matrix with exponential scaling. This yields a trajectory that spirals out in the case where $\mathrm{Re}(\lambda) = a > 0$ (look to the original vector field to see whether it's clockwise or counterclockwise), or a trajectory that spirals inward toward **0** in the case where $\mathrm{Re}(\lambda) = a < 0$.

To derive this expression for $[e^{t\mathbf{B}}]$, make another coordinate change with complex eigenvectors starting with $\mathbf{B} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. **B** will have the same eigenvalues of **A**, namely $\lambda = a + ib$ and $\bar{\lambda} = a - ib$, and $\lambda\mathbf{I} - \mathbf{B} = \begin{bmatrix} \lambda - a & b \\ -b & \lambda - a \end{bmatrix}$. Using the eigenvalue $\lambda = a + ib$, we seek a complex eigenvector $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ such that $\begin{bmatrix} ib & b \\ -b & ib \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} ib\alpha + b\beta \\ -b\alpha + ib\beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. This implies that $\alpha = i\beta$, so one such complex eigenvector is $\mathbf{w} = \begin{bmatrix} i \\ 1 \end{bmatrix}$. The eigenvalue $\bar{\lambda} = a - ib$ will then give eigenvector $\hat{\mathbf{w}} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$. Using the (complex) change of basis matrix $\mathbf{P} = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$, we have that $\mathbf{P}^{-1}\mathbf{B}\mathbf{P} = \mathbf{D} = \begin{bmatrix} a+ib & 0 \\ 0 & a-ib \end{bmatrix}$. Just as in the case of real eigenvalues, it follows that:

$$[e^{t\mathbf{B}}] = \mathbf{P}[e^{t\mathbf{D}}]\mathbf{P}^{-1} = \frac{1}{2i}\begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^{(a+ib)t} & 0 \\ 0 & e^{(a-ib)t} \end{bmatrix}\begin{bmatrix} 1 & i \\ -1 & i \end{bmatrix} = e^{at}\begin{bmatrix} \frac{e^{ibt}+e^{-ibt}}{2} & -\frac{e^{ibt}-e^{-ibt}}{2i} \\ \frac{e^{ibt}-e^{-ibt}}{2i} & \frac{e^{ibt}+e^{-ibt}}{2} \end{bmatrix} = e^{at}\begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix}.$$

Note that the exponential factor e^{at} will grow if $a = \mathrm{Re}(\lambda) > 0$ and decay if $a = \mathrm{Re}(\lambda) < 0$. Further note that the matrix $\begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix}$ is a time-varying rotation matrix with rotational frequency *b*. The product of the exponential factor and the time-varying rotation matrix means that the trajectories associated with the evolution matrix $[e^{t\mathbf{B}}]$ will be either outward or inward spirals depending upon whether $a > 0$ or $a < 0$. In the case where $a = 0$ we would get closed (periodic) trajectories – circles, in fact, for this standard case.

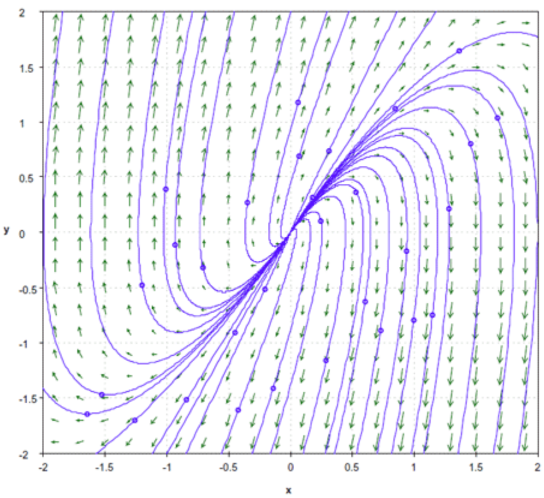
These calculations enable us to write down a closed form expression for the solution of this linear system, namely $\mathbf{x}(t) = [e^{t\mathbf{A}}]\mathbf{x}(0)$ where $[e^{t\mathbf{A}}] = \mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1} = e^{at}\mathbf{S}\begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix}\mathbf{S}^{-1}$. However, the more important result is the ability to qualitatively describe the trajectories for this system by knowing only the real part of the eigenvalues of the matrix **A** and the direction of the corresponding vector field (clockwise vs. counterclockwise).



Problem: Solve the system $\begin{cases} \dfrac{dx}{dt} = 2x - 5y \\ \dfrac{dy}{dt} = 2x - 4y \end{cases}$ with initial conditions $x(0) = 0,\ y(0) = 1$.

Solution: In matrix form, we have $\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$ where $\mathbf{A} = \begin{bmatrix} 2 & -5 \\ 2 & -4 \end{bmatrix}$ and $\mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. We again start by finding the eigenvalues of the matrix: $\lambda\mathbf{I} - \mathbf{A} = \begin{bmatrix} \lambda-2 & 5 \\ -2 & \lambda+4 \end{bmatrix}$, and the characteristic polynomial is $p_{\mathbf{A}}(\lambda) = \lambda^2 + 2\lambda + 2 = (\lambda+1)^2 + 1$. This gives the complex eigenvalue pair $\lambda = -1 + i$ and $\bar{\lambda} = -1 - i$. We seek a complex eigenvector for the first of these: $\begin{bmatrix} -3+i & 5 \\ -2 & 3+i \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ gives the (redundant) equations $(-3+i)\alpha + 5\beta = 0$ and $-2\alpha + (3+i)\beta = 0$. The first of these can be written as $5\beta = (3-i)\alpha$, and an easy solution to this is where $\alpha = 5,\ \beta = 3 - i$. (We could also have used the second equation – which is a scalar multiple of the first. The eigenvector might then have been different, but ultimately we'll get the same result.) This gives the complex eigenvector $\mathbf{w} = \begin{bmatrix} 5 \\ 3-i \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix} + i\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \mathbf{u} + i\mathbf{v}$. We have shown that with the specially chosen basis $\mathscr{B} = \{\mathbf{v}, \mathbf{u}\}$, the new system will have standard matrix $[\mathbf{A}]_{\mathscr{B}} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \mathbf{B}$ where *a* is the real part of the complex eigenvalue and b is its imaginary part. We also showed that $[e^{t\mathbf{B}}] = e^{at}\begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix}$. In this example, $a = -1$ and $b = 1$, $\mathbf{S} = [\mathbf{v} \;\; \mathbf{u}] = \begin{bmatrix} 0 & 5 \\ -1 & 3 \end{bmatrix}$, $\mathbf{S}^{-1} = \frac{1}{5}\begin{bmatrix} 3 & -5 \\ 1 & 0 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$, and $[e^{t\mathbf{B}}] = e^{-t}\begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$. The solution to the system is therefore $\mathbf{x}(t) = [e^{t\mathbf{A}}] = \mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1}\mathbf{x}(0) = \frac{e^{-t}}{5}\begin{bmatrix} 0 & 5 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}\begin{bmatrix} 3 & -5 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$= \frac{e^{-t}}{5}\begin{bmatrix} 5\sin t & 5\cos t \\ -\cos t + 3\sin t & \sin t + 3\cos t \end{bmatrix}\begin{bmatrix} -5 \\ 0 \end{bmatrix} = e^{-t}\begin{bmatrix} -5\sin t \\ \cos t - 3\sin t \end{bmatrix}$. That is, $\begin{Bmatrix} x(t) = -5e^{-t}\sin t \\ y(t) = e^{-t}(\cos t - 3\sin t) \end{Bmatrix}$.



Repeated eigenvalue case [with geometric multiplicity (GM) less than the algebraic multiplicity (AM)]:

Problem: Solve the system $\begin{Bmatrix} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -4x + 4y \end{Bmatrix}$ with initial conditions $x(0) = 3,\ y(0) = 2$.

Solution: In matrix form, we have $\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$ where $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix}$ and $\mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$. We again start by finding the eigenvalues of the matrix: $\lambda\mathbf{I} - \mathbf{A} = \begin{bmatrix} \lambda & -1 \\ 4 & \lambda - 4 \end{bmatrix}$, and the characteristic polynomial is $p_{\mathbf{A}}(\lambda) = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2$. This gives the repeated eigenvalue $\lambda = 2$ with (algebraic) multiplicity 2. We seek eigenvectors: $\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ gives the (redundant) equations $2\alpha - \beta = 0$ and $4\alpha - 2\beta = 0$. Therefore $\beta = 2\alpha$, so we can choose $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ or any scalar multiple of this as an eigenvector, but we are unable to find a second linearly independent eigenvector. (We say that the geometric multiplicity of the $\lambda = 2$ eigenvalue is 1.)

The standard procedure in this case is to seek a ***generalized eigenvector*** for this repeated eigenvalue, i.e. a vector $\mathbf{v}_2$ such that $(\lambda\mathbf{I} - \mathbf{A})\mathbf{v}_2$ is not zero, but rather a multiple of the eigenvector $\mathbf{v}_1$. Specifically, we seek a vector such that $\mathbf{A}\mathbf{v}_2 = \mathbf{v}_1 + \lambda\mathbf{v}_2$. This translates into seeking $\mathbf{v}_2$ such that $(\lambda\mathbf{I} - \mathbf{A})\mathbf{v}_2 = -\mathbf{v}_1$. That is, $\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$. This gives redundant equations the first of which is $2\alpha - \beta = -1$ or $\beta = 2\alpha + 1$. If we (arbitrarily) choose $\alpha = 0$, then $\beta = 1$, so $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The fact that $\begin{Bmatrix} \mathbf{A}\mathbf{v}_1 = 2\mathbf{v}_1 \\ \mathbf{A}\mathbf{v}_2 = \mathbf{v}_1 + 2\mathbf{v}_2 \end{Bmatrix}$ tells us that with the change of basis matrix $\mathbf{S} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, we will have $[\mathbf{A}]_{\mathcal{B}} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} = \mathbf{B}$.

The standard form in this repeated eigenvalue case is a matrix of the form $\mathbf{B} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$. (There are analogous forms in cases larger than 2×2 matrices.) Note that we can write $\mathbf{B} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} = \lambda\mathbf{I} + \mathbf{P}$ where $\mathbf{P} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

There is a simple relationship between the solutions of the systems $\frac{d\mathbf{x}}{dt} = \mathbf{Bx}$ and $\frac{d\mathbf{u}}{dt} = \mathbf{Pu}$, namely $\mathbf{x}(t) = e^{\lambda t}\mathbf{u}(t)$. This is easily seen by differentiation:

$$\frac{d\mathbf{x}}{dt} = \frac{d}{dt}[e^{\lambda t}\mathbf{u}(t)] = e^{\lambda t}\frac{d\mathbf{u}}{dt} + \lambda e^{\lambda t}\mathbf{u} = e^{\lambda t}\mathbf{Pu} + \lambda e^{\lambda t}\mathbf{u} = e^{\lambda t}(\mathbf{Pu} + \lambda\mathbf{Iu}) = e^{\lambda t}(\lambda\mathbf{I} + \mathbf{P})\mathbf{u} = (\lambda\mathbf{I} + \mathbf{P})e^{\lambda t}\mathbf{u} = \mathbf{Bx}$$

together with the fact that $\mathbf{x}(0) = \mathbf{u}(0)$. Furthermore, solving $\frac{d\mathbf{u}}{dt} = \mathbf{Pu}$ is simple. If $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, then with the matrix $\mathbf{P} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ we have $\begin{Bmatrix} u_1'(t) = u_2 \\ u_2'(t) = 0 \end{Bmatrix}$. The second equation gives that $u_2(t) = c_2 = u_2(0)$, a constant. The first equation is then $u_1'(t) = u_2(0)$, so $u_1(t) = u_2(0)\cdot t + c_1$. At $t = 0$ this gives $u_1(0) = c_1$, so $u_1(t) = u_1(0) + u_2(0)\cdot t$. Together this gives:

$$\mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} u_1(0) + u_2(0)\cdot t \\ u_2(0) \end{bmatrix} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\begin{bmatrix} u_1(0) \\ u_2(0) \end{bmatrix} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\mathbf{u}(0) = \begin{bmatrix} e^{t\mathbf{P}} \end{bmatrix}\mathbf{u}(0)$$

Therefore $\mathbf{x}(t) = e^{\lambda t}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\mathbf{x}(0) = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix}\mathbf{x}(0)$, so $\begin{bmatrix} e^{t\mathbf{B}} \end{bmatrix} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix}$ for $\mathbf{B} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$.

If we apply this to the problem at hand, we get $[e^{t\mathbf{B}}] = \begin{bmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{bmatrix}$. The solution to the system is therefore

$$\mathbf{x}(t) = [e^{t\mathbf{A}}] = \mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1}\mathbf{x}(0) = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} e^{2t} & te^{2t} \\ 2e^{2t} & 2te^{2t} + e^{2t} \end{bmatrix}\begin{bmatrix} 3 \\ -4 \end{bmatrix}$$

$$= \begin{bmatrix} 3e^{2t} - 4te^{2t} \\ 6e^{2t} - 8te^{2t} - 4e^{2t} \end{bmatrix} = \begin{bmatrix} 3e^{2t} - 4te^{2t} \\ 2e^{2t} - 8te^{2t} \end{bmatrix} = e^{2t}\begin{bmatrix} 3 - 4t \\ 2 - 8t \end{bmatrix}$$. That is, $\begin{Bmatrix} x(t) = e^{2t}(3 - 4t) \\ y(t) = e^{2t}(2 - 8t) \end{Bmatrix}$.

It's worth noting that this can also be expressed as $\mathbf{x}(t) = e^{2t}\begin{bmatrix} 3 \\ 2 \end{bmatrix} - 4te^{2t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

The phase portrait in this case has just one invariant (eigenvector) direction. It gives an unstable **node** which can be viewed as a degenerate case of a (clockwise) outward spiral that cannot get past the eigenvector direction.

Moral of the Story: It's always possible to find a special basis relative to which a given linear system is in its simplest possible form. The new basis provides a way to decompose the given problem into several simple, standard problems which can be easily solved. Any complication in the algebraic expressions for the solution is the result of changing back to the original coordinates.

The standard 2×2 cases are:

Diagonalizable with eigenvalues λ_1, λ_2: $\mathbf{B} = \mathbf{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ $\quad [e^{t\mathbf{B}}] = [e^{t\mathbf{D}}] = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$

Complex pair of eigenvalues $\lambda = a \pm ib$: $\mathbf{B} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ $\quad [e^{t\mathbf{B}}] = e^{at}\begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix}$

Repeated eigenvalue λ with $\mathrm{GM} < \mathrm{AM}$: $\mathbf{B} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ $\quad [e^{t\mathbf{B}}] = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix}$

In general, you should expect to encounter systems more complicated than these 2×2 examples. To illustrate the line of reasoning in a significantly more complicated case, here is a **Big Problem**.

Big Problem: a) Find the general solution for the following system of differential equations:

$$\left\{\begin{aligned} \frac{dx_1}{dt} &= 2x_1 - 4x_4 + 3x_5 \\ \frac{dx_2}{dt} &= 2x_2 - 2x_3 + 2x_4 \\ \frac{dx_3}{dt} &= x_2 - x_4 \\ \frac{dx_4}{dt} &= -x_4 \\ \frac{dx_5}{dt} &= -3x_4 + 2x_5 \end{aligned}\right\}$$

b) Find the solution in the case where $\mathbf{x}(0) = \begin{bmatrix} 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$.

Solution: This is a continuous dynamical system of the form $\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$ where $\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 & -4 & 3 \\ 0 & 2 & -2 & 2 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 & 2 \end{bmatrix}$.

We start by seeking the eigenvalues. We have $\lambda\mathbf{I} - \mathbf{A} = \begin{bmatrix} \lambda-2 & 0 & 0 & 4 & -3 \\ 0 & \lambda-2 & 2 & -2 & 0 \\ 0 & -1 & \lambda & 1 & 0 \\ 0 & 0 & 0 & \lambda+1 & 0 \\ 0 & 0 & 0 & 3 & \lambda-2 \end{bmatrix}$.

The characteristic polynomial is $p_{\mathbf{A}}(\lambda) = (\lambda-2)^2(\lambda+1)(\lambda^2 - 2\lambda + 2)$ which yields the repeated eigenvalue $\lambda_1 = \lambda_2 = 2$ (with algebraic multiplicity 2), the distinct eigenvalue $\lambda_3 = -1$, and the complex pair $\lambda_4 = 1+i$ and $\lambda_5 = \overline{\lambda}_4 = 1-i$.

The repeated eigenvalue $\lambda_1 = \lambda_2 = 2$ yields just one eigenvector $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, so its geometric multiplicity if just 1.

We then seek a "generalized eigenvector" $\mathbf{v}_2$ such that $\mathbf{A}\mathbf{v}_2 = \mathbf{v}_1 + \lambda\mathbf{v}_2$ where $\lambda = 2$. That is, we seek a vector $\mathbf{v}_2$ such that $\lambda\mathbf{v}_2 - \mathbf{A}\mathbf{v}_2 = (\lambda\mathbf{I} - \mathbf{A})\mathbf{v}_2 = -\mathbf{v}_1$. This is just an inhomogeneous system which yields solutions of the form $\mathbf{v}_2 = \begin{bmatrix} t \\ 0 \\ 0 \\ 0 \\ 1/3 \end{bmatrix}$. For simplicity, take the solution with $t = 0$, i.e. $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1/3 \end{bmatrix}$.

The eigenvalue $\lambda_3 = -1$ yields the eigenvector $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 3 \\ 3 \end{bmatrix}$. A straightforward calculation with the complex eigenvalue $\lambda_4 = 1+i$ yields the complex eigenvector $\mathbf{v} = \begin{bmatrix} 0 \\ 1+i \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{v}_5 + i\,\mathbf{v}_4$ in accordance with the method previously derived.

Using the basis $\mathcal{B} = \left\{ \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1/3 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 3 \\ 3 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_5 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ and change of basis matrix

$\mathbf{S} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 1/3 & 3 & 0 & 0 \end{bmatrix}$, we compute the inverse matrix $\mathbf{S}^{-1} = \begin{bmatrix} 1 & 0 & 0 & -1/3 & 0 \\ 0 & 0 & 0 & -3 & 3 \\ 0 & 0 & 0 & 1/3 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix}$.

We know that $\begin{cases} \mathbf{A}\mathbf{v}_1 = 2\mathbf{v}_1 \\ \mathbf{A}\mathbf{v}_2 = \mathbf{v}_1 + 2\mathbf{v}_2 \\ \mathbf{A}\mathbf{v}_3 = -\mathbf{v}_3 \\ \mathbf{A}\mathbf{v}_4 = \mathbf{v}_4 + \mathbf{v}_5 \\ \mathbf{A}\mathbf{v}_5 = -\mathbf{v}_4 + \mathbf{v}_5 \end{cases}$, so the matrix of $\mathbf{A}$ relative to the basis $\mathcal{B}$ is

$$\mathbf{B} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \left[\begin{array}{cc|c|cc} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right].$$

Since $\mathbf{A} = \mathbf{S}\mathbf{B}\mathbf{S}^{-1}$, it will be the case that the evolution matrices are related via $\left[e^{t\mathbf{A}}\right] = \mathbf{S}\left[e^{t\mathbf{B}}\right]\mathbf{S}^{-1}$ where

$$\left[e^{t\mathbf{B}}\right] = \left[\begin{array}{cc|c|cc} e^{2t} & te^{2t} & 0 & 0 & 0 \\ 0 & e^{2t} & 0 & 0 & 0 \\ \hline 0 & 0 & e^{-t} & 0 & 0 \\ \hline 0 & 0 & 0 & e^t \cos t & -e^t \sin t \\ 0 & 0 & 0 & e^t \sin t & e^t \cos t \end{array}\right].$$

The solution is then

$$\mathbf{x}(t) = \left[e^{t\mathbf{A}}\right]\mathbf{x}(0) = \mathbf{S}\left[e^{t\mathbf{B}}\right]\mathbf{S}^{-1}\mathbf{x}(0) = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 1/3 & 3 & 0 & 0 \end{bmatrix} \left[\begin{array}{cc|c|cc} e^{2t} & te^{2t} & 0 & 0 & 0 \\ 0 & e^{2t} & 0 & 0 & 0 \\ \hline 0 & 0 & e^{-t} & 0 & 0 \\ \hline 0 & 0 & 0 & e^t \cos t & -e^t \sin t \\ 0 & 0 & 0 & e^t \sin t & e^t \cos t \end{array}\right] \begin{bmatrix} 1 & 0 & 0 & -1/3 & 0 \\ 0 & 0 & 0 & -3 & 3 \\ 0 & 0 & 0 & 1/3 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix} \mathbf{x}(0).$$

If we multiply the leftmost matrices and write $\mathbf{S}^{-1}\mathbf{x}(0)=\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix}$, this yields the general solution:

$$\mathbf{x}(t)=\left[e^{t\mathbf{A}}\right]\mathbf{x}(0)=\mathbf{S}\left[e^{t\mathbf{B}}\right]\mathbf{S}^{-1}\mathbf{x}(0)=\left[\begin{array}{cc:c:cc} e^{2t} & te^{2t} & e^{-t} & 0 & 0 \\ 0 & 0 & 0 & e^t(\cos t+\sin t) & e^t(\cos t-\sin t) \\ \hdashline 0 & 0 & 3e^{-t} & e^t\sin t & e^t\cos t \\ \hdashline 0 & 0 & 3e^{-t} & 0 & 0 \\ 0 & \frac{1}{3}e^{2t} & 3e^{-t} & 0 & 0 \end{array}\right]\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix}$$

$$\text{or }\left\{\begin{array}{l} x_1(t)=c_1e^{2t}+c_2te^{2t}+c_3e^{-t} \\ x_2(t)=c_4e^t(\cos t+\sin t)+c_5e^t(\cos t-\sin t) \\ x_3(t)=3c_3e^{-t}+c_4e^t\sin t+c_5e^t\cos t \\ x_4(t)=3c_3e^{-t} \\ x_5(t)=\frac{1}{3}c_2e^{2t}+3c_3e^{-t} \end{array}\right\}.$$

If, on the other hand, we use the initial condition $\mathbf{x}(0)=\begin{bmatrix} 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$, we get the specific solution:

$$\mathbf{x}(t)=\left[\begin{array}{cc:c:cc} e^{2t} & te^{2t} & e^{-t} & 0 & 0 \\ 0 & 0 & 0 & e^t(\cos t+\sin t) & e^t(\cos t-\sin t) \\ \hdashline 0 & 0 & 3e^{-t} & e^t\sin t & e^t\cos t \\ \hdashline 0 & 0 & 3e^{-t} & 0 & 0 \\ 0 & \frac{1}{3}e^{2t} & 3e^{-t} & 0 & 0 \end{array}\right]\begin{bmatrix} 13/3 \\ -3 \\ 2/3 \\ 3 \\ 1 \end{bmatrix}$$

$$\text{or }\left\{\begin{array}{l} x_1(t)=\frac{13}{3}e^{2t}-3te^{2t}+\frac{2}{3}e^{-t} \\ x_2(t)=e^t(4\cos t+2\sin t) \\ x_3(t)=2e^{-t}+e^t(3\sin t+\cos t) \\ x_4(t)=2e^{-t} \\ x_5(t)=-e^{2t}+2e^{-t} \end{array}\right\}.$$