Exam #2 Reference Sheet

Fourier coefficients for periodic functions of period 2π :

$$f(t)=\frac{a_0}{2}+a_1\cos t+b_1\sin t+a_2\cos 2t+b_2\sin 2t+\cdots+a_n\cos nt+b_n\sin nt+\cdots$$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\,dt\,,\quad a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\cos nt\,dt\,,\quad b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\sin nt\,dt$$

For example, if $\mathrm{sq}(t)$ is the odd function of period 2π which has value 1 between 0 and π, then

$$\mathrm{sq}(t)=\frac{4}{\pi}\left(\sin t+\frac{\sin 3t}{3}+\frac{\sin 5t}{5}+\cdots\right)=\frac{4}{\pi}\sum_{n\text{ odd}}\frac{\sin nt}{n}$$ [square-wave function]

Properties of the Laplace transform

0. Definition: $\mathcal{L}[f(t)]=F(s)=\int_{0-}^{\infty}e^{-st}f(t)dt$ for $\mathrm{Re}(s)\gg 0$.

1. Linearity: $\mathcal{L}[af(t)+bg(t)]=a\mathcal{L}[f(t)]+b\mathcal{L}[g(t)]=aF(s)+bG(s)$.

2. Inverse transform: $F(s)$ essentially determines $f(t)$.

3. s-shift rule: $\mathcal{L}[e^{rt}f(t)]=F(s-r)$.

4. t-shift rule: $\mathcal{L}[f(t-a)]=e^{-as}F(s)$ if $a\ge 0$ and $f(t)=0$ for $t<0$.

   This may also be expressed as $\mathcal{L}[f_a(t)]=e^{-as}F(s)$ where $f_a(t)=u(t-a)f(t-a)=\begin{cases}f(t-a) & \text{if } t>a\\ 0 & \text{if } t<a\end{cases}$.

5. s-derivative rule: $\mathcal{L}[t\,f(t)]=-F'(s)$.

6. t-derivative rule: $\mathcal{L}[f'(t)]=sF(s)-f(0-)$

   $\mathcal{L}[f''(t)]=s^2F(s)-sf(0-)-f'(0-)$

   $\mathcal{L}[f^{(n)}(t)]=s^nF(s)-s^{n-1}f(0-)-s^{n-2}f'(0-)-\cdots-f^{(n-1)}(0-)$

7. Convolution rules: $\mathcal{L}[f(t)*g(t)]=F(s)G(s)$, $(f*g)(t)=\int_0^t f(t-\tau)g(\tau)d\tau$, $(f*g)(t)=(g*f)(t)$.

8. Weight function: $\mathcal{L}[w(t)]=W(s)$, $w(t)$ the unit impulse response.

   [If $\delta(t)$ is regarded as the input signal in $p(D)x=\delta(t)$ with rest initial conditions, $W(s)=\frac{1}{p(s)}$.]

Formulas for the Laplace transform

$\mathcal{L}[1]=\frac{1}{s}$

$\mathcal{L}[\delta(t)]=1$

$\mathcal{L}[\delta(t-a)]=e^{-as}$

$\mathcal{L}[e^{at}]=\frac{1}{s-a}$

$\mathcal{L}[t]=\frac{1}{s^2}$

$\mathcal{L}[t^n]=\frac{n!}{s^{n+1}}$

$\mathcal{L}[\cos(\omega t)]=\frac{s}{s^2+\omega^2}$

$\mathcal{L}[\sin(\omega t)]=\frac{\omega}{s^2+\omega^2}$

$\mathcal{L}[t^n f(t)]=(-1)^nF^{(n)}(s)$

$\mathcal{L}[u(t-a)f(t-a)]=e^{-as}F(s)$

$\mathcal{L}[u(t-a)f(t)]=e^{-as}\mathcal{L}[f(t+a)]$

$\mathcal{L}[t\cos(\omega t)]=\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$

$\mathcal{L}[t\sin(\omega t)]=\frac{2\omega s}{(s^2+\omega^2)^2}$

$\mathcal{L}[e^{zt}\cos(\omega t)]=\frac{s-z}{(s-z)^2+\omega^2}$

$\mathcal{L}[e^{zt}\sin(\omega t)]=\frac{\omega}{(s-z)^2+\omega^2}$