

Math E-21c – Continuous Dynamical Systems – Part 2

Solving systems using diagonalization and evolution matrices

Given an $n \times n$ matrix \mathbf{A} , suppose \mathbf{S} is a change of basis matrix corresponding to either diagonalization or reduction to Jordan Canonical Form. We will have $\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{B}$ in this case, where \mathbf{B} is diagonal or otherwise in simplest form. We then calculate $\mathbf{A} = \mathbf{S}\mathbf{B}\mathbf{S}^{-1}$, and substitution gives $\frac{d\mathbf{x}}{dt} = \mathbf{S}\mathbf{B}\mathbf{S}^{-1}\mathbf{x}$.

Multiplying on the left by \mathbf{S}^{-1} and using the basic calculus fact that $\frac{d}{dt}(\mathbf{M}\mathbf{x}) = \mathbf{M}\frac{d\mathbf{x}}{dt}$ for any (constant) matrix \mathbf{M} , we have $\mathbf{S}^{-1}\frac{d\mathbf{x}}{dt} = \frac{d(\mathbf{S}^{-1}\mathbf{x})}{dt} = \mathbf{B}(\mathbf{S}^{-1}\mathbf{x})$. If we write $\mathbf{u} = \mathbf{S}^{-1}\mathbf{x} = [\mathbf{x}]_{\mathcal{B}}$, where \mathcal{B} is the new, preferred basis, then in these new coordinates the system becomes $\frac{d\mathbf{u}}{dt} = \mathbf{B}\mathbf{u}$, but now the system will be much more straightforward to solve.

The diagonalizable case

In the case where \mathbf{B} is a diagonal matrix with the eigenvalues of \mathbf{A} on the diagonal, the system is just

$$\frac{d\mathbf{u}}{dt} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \mathbf{u} \text{ or } \left\{ \begin{array}{l} \frac{du_1}{dt} = \lambda_1 u_1 \\ \vdots \\ \frac{du_n}{dt} = \lambda_n u_n \end{array} \right\}.$$

This has the solution $\left\{ \begin{array}{l} u_1(t) = e^{\lambda_1 t} u_1(0) \\ \vdots \\ u_n(t) = e^{\lambda_n t} u_n(0) \end{array} \right\}$ or $\mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} u_1(0) \\ \vdots \\ u_n(0) \end{bmatrix} = [e^{t\mathbf{B}}]\mathbf{u}(0)$.

To revert back to the original coordinates, we write $\mathbf{x} = \mathbf{S}\mathbf{u}$, so $\mathbf{x}(t) = \mathbf{S}\mathbf{u}(t) = \mathbf{S}[e^{t\mathbf{B}}]\mathbf{u}(0) = \mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1}\mathbf{x}(0)$. If we denote the evolution matrix for the system in its original coordinates as $[e^{t\mathbf{A}}]$ where $\mathbf{x}(t) = [e^{t\mathbf{A}}]\mathbf{x}(0)$, then the previous calculation gives the simple relation $[e^{t\mathbf{A}}] = \mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1}$.

In other words, the evolution matrices for the solution are in the same relationship as the matrices \mathbf{A} and \mathbf{B} , namely $\mathbf{A} = \mathbf{S}\mathbf{B}\mathbf{S}^{-1}$. This pattern is very easy to remember, and this same pattern will again be the case where \mathbf{B} is not diagonal but where the corresponding evolution matrix is still relatively easy to calculate.

$$\mathbf{A} = \mathbf{S}\mathbf{B}\mathbf{S}^{-1} \Rightarrow [e^{t\mathbf{A}}] = \mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1}, \text{ and the solution of the original system will be } \boxed{\mathbf{x}(t) = [e^{t\mathbf{A}}]\mathbf{x}(0)}.$$

The complex eigenvalue case

Suppose we want to solve a system of the form $\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$ where \mathbf{A} is an 2×2 real matrix with a complex conjugate pair of eigenvalues $\lambda = a + ib$ and $\lambda = a - ib$. There are several reasonable ways to proceed, but they all come down to determining the evolution matrix $[e^{t\mathbf{A}}]$ so that we can solve for $\mathbf{x}(t) = [e^{t\mathbf{A}}]\mathbf{x}(0)$.

First, put the system into (real) normal form.

Use the complex eigenvalue $\lambda = a + ib$ to find a complex eigenvector $\mathbf{w} = \mathbf{u} + i\mathbf{v}$. If we change to the basis $\{\mathbf{v}, \mathbf{u}\}$ then, using the change of basis matrix $\mathbf{S} = [\mathbf{v} \ \mathbf{u}]$, we'll get $\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{B} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, a rotation-dilation matrix. Noting, as before, that $\mathbf{A} = \mathbf{S}\mathbf{B}\mathbf{S}^{-1} \Rightarrow [e^{t\mathbf{A}}] = \mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1}$, we need only to determine $[e^{t\mathbf{B}}]$.

Second, find the evolution matrix for the (real) normal form.

In fact, if $\mathbf{B} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ then $[e^{\mathbf{B}t}] = e^{at} \begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix}$, a time-varying rotation matrix with exponential scaling. For any initial condition (except the zero vector), this yields a trajectory that spirals out in the case where $\text{Re}(\lambda) = a > 0$ (look to the original vector field to see whether it's clockwise or counterclockwise), or a trajectory that spirals inward toward $\mathbf{0}$ in the case where $\text{Re}(\lambda) = a < 0$.

To derive this expression for $[e^{\mathbf{B}t}]$, make another coordinate change with complex eigenvectors starting with $\mathbf{B} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. We know this has the same eigenvalues of \mathbf{A} , namely $\lambda = a + ib$ and $\bar{\lambda} = a - ib$. Use $\lambda = a + ib$ to get the complex eigenvector $\mathbf{w} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$. The eigenvalue $\bar{\lambda} = a - ib$ will then give the eigenvector $\bar{\mathbf{w}} = \begin{bmatrix} 1 \\ i \end{bmatrix}$. Using the (complex) change of basis matrix $\mathbf{P} = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}$, we have that $\mathbf{P}^{-1}\mathbf{B}\mathbf{P} = \mathbf{D} = \begin{bmatrix} a + ib & 0 \\ 0 & a - ib \end{bmatrix}$. It follows that (using Euler's Formula as needed):

$$[e^{\mathbf{B}t}] = \mathbf{P}[e^{\mathbf{D}t}]\mathbf{P}^{-1} = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \begin{bmatrix} e^{(a+ib)t} & 0 \\ 0 & e^{(a-ib)t} \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} = e^{at} \begin{bmatrix} \frac{e^{ibt} + e^{-ibt}}{2} & -\frac{e^{ibt} - e^{-ibt}}{2i} \\ \frac{e^{ibt} - e^{-ibt}}{2i} & \frac{e^{ibt} + e^{-ibt}}{2} \end{bmatrix} = e^{at} \begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix}.$$

These calculations enable us to write down a closed form expression for the solution of this linear system, namely $\mathbf{x}(t) = [e^{t\mathbf{A}}]\mathbf{x}(0)$ where $[e^{t\mathbf{A}}] = \mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1} = e^{at}\mathbf{S} \begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix} \mathbf{S}^{-1}$. However, the more important result is the ability to qualitatively describe the trajectories for this system by knowing only the real part of the eigenvalues of the matrix \mathbf{A} and the direction of the corresponding vector field (clockwise vs. counterclockwise).

Repeated eigenvalues (with geometric multiplicity less than the algebraic multiplicity)

Suppose we want to solve a system of the form $\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$ where \mathbf{A} is a non-diagonalizable 2×2 real matrix with a repeated eigenvalue λ . We've seen that in this case, we can always find a change of basis matrix \mathbf{S} such that $\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{B} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$. As in the previous two cases, $\mathbf{A} = \mathbf{S}\mathbf{B}\mathbf{S}^{-1} \Rightarrow [e^{t\mathbf{A}}] = \mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1}$ and it comes down to finding $[e^{t\mathbf{B}}]$. This is perhaps most easily done by explicitly solving the corresponding differential equations.

In the new coordinates, this system translates into $\left\{ \begin{array}{l} \frac{du_1}{dt} = \lambda u_1 + u_2 \\ \frac{du_2}{dt} = \lambda u_2 \end{array} \right\}$. The second equation is easily solved to get

$u_2(t) = e^{\lambda t} u_2(0)$. We can guess a solution for the first equation of the form $u_1(t) = c_1 t e^{\lambda t} + c_2 e^{\lambda t}$. Differentiating this and substituting into the first equation, we get $c_1(e^{\lambda t} + \lambda t e^{\lambda t}) + c_2 \lambda e^{\lambda t} = \lambda(c_1 t e^{\lambda t} + c_2 e^{\lambda t}) + e^{\lambda t} u_2(0)$.

Comparing like terms, we conclude that $c_1 = u_2(0)$. Substituting $t = 0$, we further conclude that $u_1(0) = c_2$.

Putting these results together, we get $u_1(t) = u_2(0) t e^{\lambda t} + u_1(0) e^{\lambda t} = e^{\lambda t} u_1(0) + t e^{\lambda t} u_2(0)$. We therefore have that

$$\mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} e^{\lambda t} u_1(0) + t e^{\lambda t} u_2(0) \\ e^{\lambda t} u_2(0) \end{bmatrix} = \begin{bmatrix} e^{\lambda t} & t e^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix} \begin{bmatrix} u_1(0) \\ u_2(0) \end{bmatrix} = \begin{bmatrix} e^{\lambda t} & t e^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix} \mathbf{u}(0)$$

So, $[e^{t\mathbf{B}}] = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix}$ in this case and the solution is given by $\mathbf{x}(t) = \mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1} = \mathbf{S} \begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix} \mathbf{S}^{-1}\mathbf{x}(0)$.

An alternate method of deriving this result may be found in the homework exercises.

Similar calculations enable us to deal with cases such as a repeated eigenvalue where the geometric multiplicity is 1 and the algebraic multiplicity is 3 (or even worse).

Finally, an actual system may exhibit several of these qualities – one or more complex pairs of eigenvalues, repeated eigenvalues, and distinct real eigenvalues. The Jordan Canonical Form of the matrix for such a system can be analyzed block by block and each of the above solutions applied within each block to determine the evolution matrix for the entire system.

The Main Idea:

Given a system of 1st order linear differential equations $\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$ with initial conditions $\mathbf{x}(0)$, we use eigenvalue-eigenvector analysis to find an appropriate basis $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ for \mathbf{R}^n and a change of basis

matrix $\mathbf{S} = \begin{bmatrix} \uparrow & & \uparrow \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ \downarrow & & \downarrow \end{bmatrix}$ such that in coordinates relative to this basis ($\mathbf{u} = \mathbf{S}^{-1}\mathbf{x}$) the system is in a standard

form with a known solution. Specifically, we find a standard matrix $\mathbf{B} = [\mathbf{A}]_{\mathcal{B}} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$, transform the system into $\frac{d\mathbf{u}}{dt} = \mathbf{B}\mathbf{u}$, solve it as $\mathbf{u}(t) = [e^{t\mathbf{B}}]\mathbf{u}(0)$ where $[e^{t\mathbf{B}}]$ is the *evolution matrix* for \mathbf{B} , then transform back to the original coordinates to get $\mathbf{x}(t) = [e^{t\mathbf{A}}]\mathbf{x}(0)$ where $[e^{t\mathbf{A}}] = \mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1}$ is the *evolution matrix* for \mathbf{B} . That is $\mathbf{x}(t) = [e^{t\mathbf{A}}] = \mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1}\mathbf{x}(0)$. This is easier to do than it is to explain, so here are a few illustrative examples:

The diagonalizable case

Problem: Solve the system $\begin{cases} \frac{dx}{dt} = 5x - 6y \\ \frac{dy}{dt} = 3x - 4y \end{cases}$ with

initial conditions $x(0) = 3, y(0) = 1$.

Solution: In matrix form, we have $\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$ where

$\mathbf{A} = \begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix}$ and $\mathbf{x}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. We start by finding

the eigenvalues of the matrix:

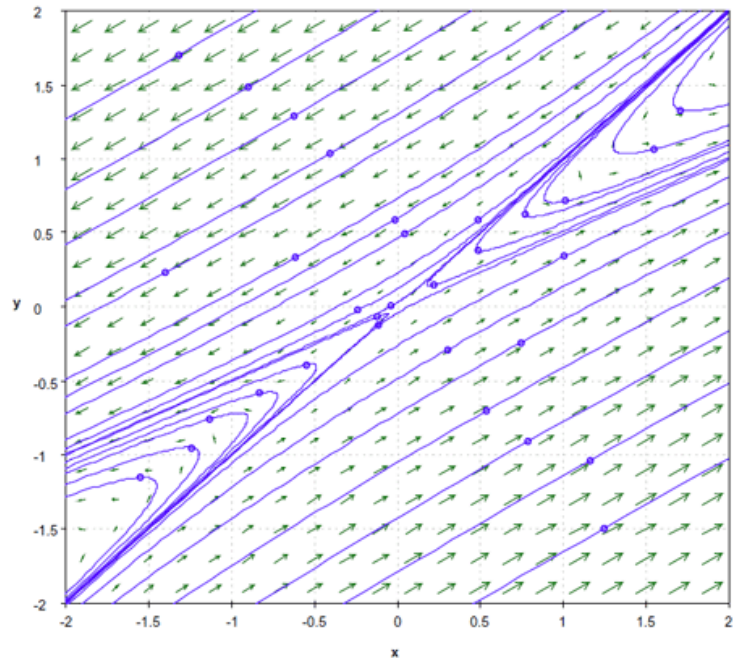
$\lambda\mathbf{I} - \mathbf{A} = \begin{bmatrix} \lambda - 5 & 6 \\ -3 & \lambda + 4 \end{bmatrix}$, and the characteristic

polynomial is $p_{\mathbf{A}}(\lambda) = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1)$.

This gives the eigenvalues $\lambda_1 = 2$ and $\lambda_2 = -1$. The

first of these gives the eigenvector $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, and the second gives the eigenvector $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. So we have

$\begin{cases} \mathbf{A}\mathbf{v}_1 = \lambda_1\mathbf{v}_1 \\ \mathbf{A}\mathbf{v}_2 = \lambda_2\mathbf{v}_2 \end{cases}$. The change of basis matrix is $\mathbf{S} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and with the new basis (of eigenvectors)



$\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ we have $[\mathbf{A}]_{\mathcal{B}} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} = \mathbf{D}$, a diagonal matrix. [There is no need to carry out the multiplication of the matrices if $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ is known to be a basis of eigenvectors. It will always yield a diagonal matrix with the eigenvalues on the diagonal.]

The evolution matrix for this diagonal matrix is $[e^{\mathbf{D}t}] = \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{bmatrix}$, and the solution of the system is:

$$\begin{aligned} \mathbf{x}(t) &= [e^{t\mathbf{A}}]\mathbf{x}(0) = \mathbf{S}[e^{\mathbf{D}t}]\mathbf{S}^{-1}\mathbf{x}(0) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2e^{2t} & e^{-t} \\ e^{2t} & e^{-t} \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 4e^{2t} - e^{-t} \\ 2e^{2t} - e^{-t} \end{bmatrix} = 2e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} - e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 2e^{2t}\mathbf{v}_1 - e^{-t}\mathbf{v}_2 \end{aligned}$$

The complex eigenvalue case

Problem: Solve the system $\begin{cases} \frac{dx}{dt} = 2x - 5y \\ \frac{dy}{dt} = 2x - 4y \end{cases}$ with

initial conditions $x(0) = 0, y(0) = 1$.

Solution: In matrix form, we have $\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$ where

$\mathbf{A} = \begin{bmatrix} 2 & -5 \\ 2 & -4 \end{bmatrix}$ and $\mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. We again start by

finding the eigenvalues of the matrix:

$\lambda\mathbf{I} - \mathbf{A} = \begin{bmatrix} \lambda - 2 & 5 \\ -2 & \lambda + 4 \end{bmatrix}$, and the characteristic

polynomial is $p_{\mathbf{A}}(\lambda) = \lambda^2 + 2\lambda + 2 = (\lambda + 1)^2 + 1$.

This gives the complex eigenvalue pair $\lambda = -1 + i$ and $\bar{\lambda} = -1 - i$. We seek a complex eigenvector

for the first of these: $\begin{bmatrix} -3+i & 5 \\ -2 & 3+i \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

gives the (redundant) equations $(-3+i)\alpha + 5\beta = 0$ and $-2\alpha + (3+i)\beta = 0$. The first of these can be written as $5\beta = (3-i)\alpha$, and an easy solution to this is where $\alpha = 5, \beta = 3-i$. (We could also have used the second equation – which is a scalar multiple of the first. The eigenvector might then have been different, but

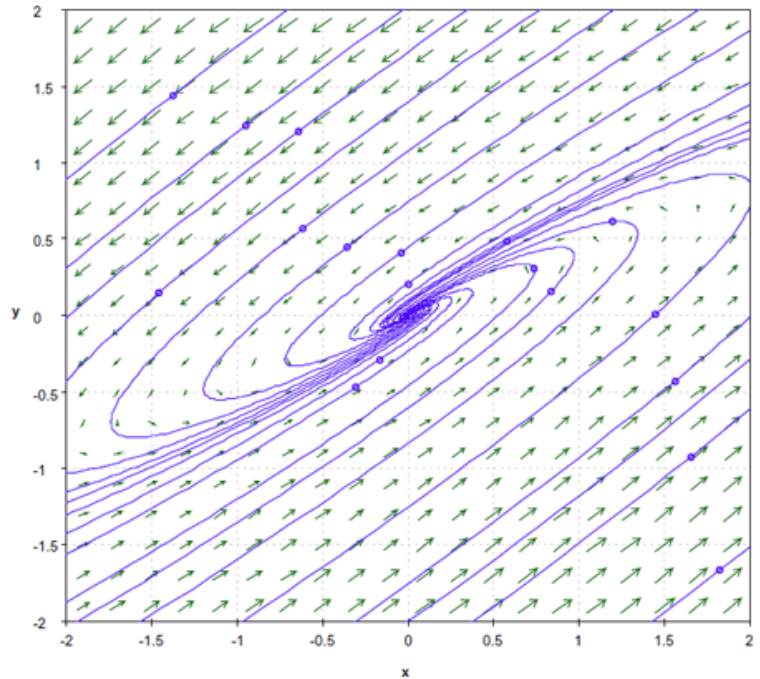
ultimately we'll get the same result.) This gives the complex eigenvector $\mathbf{w} = \begin{bmatrix} 5 \\ 3-i \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix} + i \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \mathbf{u} + i\mathbf{v}$.

We have shown that with the specially chosen basis $\mathcal{B} = \{\mathbf{v}, \mathbf{u}\}$, the new system will have standard matrix

$[\mathbf{A}]_{\mathcal{B}} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \mathbf{B}$ where a is the real part of the complex eigenvalue and b is its imaginary part.

We also showed that $[e^{t\mathbf{B}}] = e^{at} \begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix}$. In this example, $a = -1$ and $b = 1$, $\mathbf{S} = [\mathbf{v} \ \mathbf{u}] = \begin{bmatrix} 0 & 5 \\ -1 & 3 \end{bmatrix}$,

$\mathbf{S}^{-1} = \frac{1}{5} \begin{bmatrix} 3 & -5 \\ 1 & 0 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$, and $[e^{t\mathbf{B}}] = e^{-t} \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$. The solution to the system is therefore:



$$\mathbf{x}(t) = [e^{t\mathbf{A}}] = \mathbf{S}[e^{\mathbf{B}}]\mathbf{S}^{-1}\mathbf{x}(0) = \frac{e^{-t}}{5} \begin{bmatrix} 0 & 5 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} 3 & -5 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{e^{-t}}{5} \begin{bmatrix} 5 \sin t & 5 \cos t \\ -\cos t + 3 \sin t & \sin t + 3 \cos t \end{bmatrix} \begin{bmatrix} -5 \\ 0 \end{bmatrix} = e^{-t} \begin{bmatrix} -5 \sin t \\ \cos t - 3 \sin t \end{bmatrix}. \text{ That is, } \begin{cases} x(t) = -5e^{-t} \sin t \\ y(t) = e^{-t} \cos t - 3e^{-t} \sin t \end{cases}.$$

Repeated eigenvalue case [with geometric multiplicity (GM) less than the algebraic multiplicity (AM)]:

Problem: Solve the system
$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -4x + 4y \end{cases}$$

with initial conditions $x(0) = 3, y(0) = 2$.

Solution: In matrix form, we have $\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$

where $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix}$ and $\mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$. We again

start by finding the eigenvalues of the matrix:

$$\lambda\mathbf{I} - \mathbf{A} = \begin{bmatrix} \lambda & -1 \\ 4 & \lambda - 4 \end{bmatrix}, \text{ and the characteristic}$$

polynomial is $p_{\mathbf{A}}(\lambda) = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2$.

This gives the repeated eigenvalue $\lambda = 2$ with (algebraic) multiplicity 2. We seek

eigenvectors: $\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ gives the

(redundant) equations $2\alpha - \beta = 0$ and

$4\alpha - 2\beta = 0$. Therefore $\beta = 2\alpha$, so we can choose $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ or any scalar multiple of this as an eigenvector,

but we are unable to find a second linearly independent eigenvector. (We say that the geometric multiplicity of the $\lambda = 2$ eigenvalue is 1.)

The standard procedure in this case is to seek a **generalized eigenvector** for this repeated eigenvalue, i.e. a vector \mathbf{v}_2 such that $(\lambda\mathbf{I} - \mathbf{A})\mathbf{v}_2$ is not zero, but rather a multiple of the eigenvector \mathbf{v}_1 . Specifically, we seek a vector such that $\mathbf{A}\mathbf{v}_2 = \mathbf{v}_1 + \lambda\mathbf{v}_2$. This translates into seeking \mathbf{v}_2 such that $(\lambda\mathbf{I} - \mathbf{A})\mathbf{v}_2 = -\mathbf{v}_1$. That is,

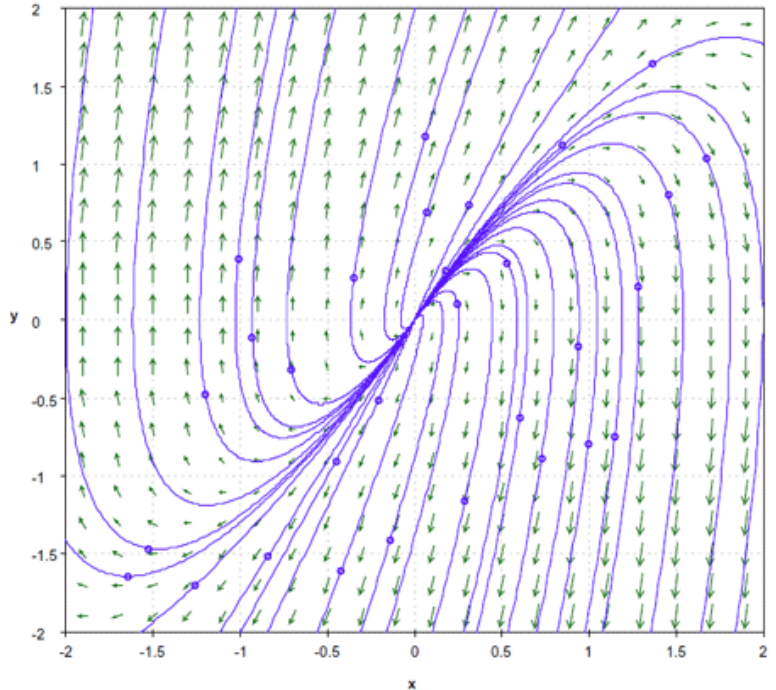
$$\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}. \text{ This gives redundant equations the first of which is } 2\alpha - \beta = -1 \text{ or } \beta = 2\alpha + 1.$$

If we (arbitrarily) choose $\alpha = 0$, then $\beta = 1$, so $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The fact that $\begin{cases} \mathbf{A}\mathbf{v}_1 = 2\mathbf{v}_1 \\ \mathbf{A}\mathbf{v}_2 = \mathbf{v}_1 + 2\mathbf{v}_2 \end{cases}$ tells us that with the

change of basis matrix $\mathbf{S} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, we will have $[\mathbf{A}]_{\mathcal{B}} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} = \mathbf{B}$.

The standard form in this repeated eigenvalue case is a matrix of the form $\mathbf{B} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$. (There are analogous

forms in cases larger than 2×2 matrices.) Note that we can write $\mathbf{B} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} = \lambda\mathbf{I} + \mathbf{P}$ where $\mathbf{P} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.



There is a simple relationship between the solutions of the systems $\frac{d\mathbf{x}}{dt} = \mathbf{B}\mathbf{x}$ and $\frac{d\mathbf{u}}{dt} = \mathbf{P}\mathbf{u}$, namely $\mathbf{x}(t) = e^{\lambda t}\mathbf{u}(t)$. This is easily seen by differentiation:

$$\frac{d\mathbf{x}}{dt} = \frac{d}{dt}[e^{\lambda t}\mathbf{u}(t)] = e^{\lambda t}\frac{d\mathbf{u}}{dt} + \lambda e^{\lambda t}\mathbf{u} = e^{\lambda t}\mathbf{P}\mathbf{u} + \lambda e^{\lambda t}\mathbf{u} = e^{\lambda t}(\mathbf{P}\mathbf{u} + \lambda\mathbf{I}\mathbf{u}) = e^{\lambda t}(\lambda\mathbf{I} + \mathbf{P})\mathbf{u} = (\lambda\mathbf{I} + \mathbf{P})e^{\lambda t}\mathbf{u} = \mathbf{B}\mathbf{x}$$

together with the fact that $\mathbf{x}(0) = \mathbf{u}(0)$.

Furthermore, solving $\frac{d\mathbf{u}}{dt} = \mathbf{P}\mathbf{u}$ is simple. If $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, then with $\mathbf{P} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ we have $\begin{cases} u_1'(t) = u_2 \\ u_2'(t) = 0 \end{cases}$.

The second equation gives that $u_2(t) = c_2 = u_2(0)$, a constant. The first equation is then $u_1'(t) = u_2(0)$, so $u_1(t) = u_2(0) \cdot t + c_1$. At $t = 0$ this gives $u_1(0) = c_1$, so $u_1(t) = u_1(0) + u_2(0) \cdot t$. Together this gives:

$$\mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} u_1(0) + u_2(0) \cdot t \\ u_2(0) \end{bmatrix} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1(0) \\ u_2(0) \end{bmatrix} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \mathbf{u}(0) = [e^{\mathbf{P}t}] \mathbf{u}(0)$$

Therefore $\mathbf{x}(t) = e^{\lambda t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \mathbf{x}(0) = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix} \mathbf{x}(0)$, so $[e^{\mathbf{B}t}] = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix}$ for $\mathbf{B} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$.

If we apply this to the problem at hand, we get $[e^{\mathbf{B}t}] = \begin{bmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{bmatrix}$. The solution to the system is therefore

$$\begin{aligned} \mathbf{x}(t) &= [e^{\mathbf{A}t}] \mathbf{x}(0) = \mathbf{S}[e^{\mathbf{B}t}]\mathbf{S}^{-1}\mathbf{x}(0) = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} e^{2t} & te^{2t} \\ 2e^{2t} & 2te^{2t} + e^{2t} \end{bmatrix} \begin{bmatrix} 3 \\ -4 \end{bmatrix} \\ &= \begin{bmatrix} 3e^{2t} - 4te^{2t} \\ 6e^{2t} - 8te^{2t} - 4e^{2t} \end{bmatrix} = \begin{bmatrix} 3e^{2t} - 4te^{2t} \\ 2e^{2t} - 8te^{2t} \end{bmatrix} = e^{2t} \begin{bmatrix} 3 - 4t \\ 2 - 8t \end{bmatrix}. \text{ That is, } \begin{cases} x(t) = e^{2t}(3 - 4t) \\ y(t) = e^{2t}(2 - 8t) \end{cases}. \end{aligned}$$

It's worth noting that this can also be expressed as $\mathbf{x}(t) = e^{2t} \begin{bmatrix} 3 \\ 2 \end{bmatrix} - 4te^{2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

The phase portrait in this case has just one invariant (eigenvector) direction. It gives an unstable **node** which can be viewed as a degenerate case of a (clockwise) outward spiral that cannot get past the eigenvector direction.

Moral of the Story: It's always possible to find a special basis relative to which a given linear system is in its simplest possible form. The new basis provides a way to decompose the given problem into several simple, standard problems which can be easily solved. Any complication in the algebraic expressions for the solution is the result of changing back to the original coordinates.

The standard 2×2 cases are:

Diagonalizable with eigenvalues λ_1, λ_2 : $\mathbf{B} = \mathbf{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ $[e^{\mathbf{B}t}] = [e^{\mathbf{D}t}] = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$

Complex pair of eigenvalues $\lambda = a \pm ib$: $\mathbf{B} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ $[e^{\mathbf{B}t}] = e^{at} \begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix}$

Repeated eigenvalue λ with GM < AM: $\mathbf{B} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ $[e^{\mathbf{B}t}] = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix} = e^{\lambda t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$

In general, you should expect to encounter systems more complicated than these 2×2 examples.