Math E-21c – Continuous Dynamical Systems – Part 1

Vector fields, Continuous Dynamical Systems, and Systems of 1st Order Linear Differential Equations

Definition: A **vector field** in $\mathbf{R}^n$ is an assignment of a vector to every point in $\mathbf{R}^n$ (with the possible exception of some singular points). This can be viewed as a function $\mathbf{F}(x_1,\cdots,x_n)=\begin{bmatrix} f_1(x_1,\cdots,x_n) \\ \vdots \\ f_n(x_1,\cdots,x_n)\end{bmatrix}$ where $f_i(x_1,\cdots,x_n)$ is the *i*-th component of the vector assigned to the point $(x_1,\cdots,x_n)$. We can also write this more succinctly as $\mathbf{F}(\mathbf{x})=\begin{bmatrix} f_1(\mathbf{x}) \\ \vdots \\ f_n(\mathbf{x})\end{bmatrix}$. In practice we usually assume some reasonable properties such as that the component functions are continuous or differentiable except perhaps at a finite number of *singular points*.

If we view the vector assigned to each point as a ***velocity vector*** associated with some smoothly varying system, a reasonable question to ask is this: Given a starting point $\mathbf{x}_0$ (the initial condition), can we find a parameterized curve $\mathbf{x}(t)$ such that $\mathbf{x}(0)=\mathbf{x}_0$ and the velocity vector at any point on this parameterized curve matches the underlying vector field, i.e. $\frac{d\mathbf{x}}{dt}=\mathbf{F}(\mathbf{x}(t))$. This is equivalent to a system of (time-independent) first-order differential equations, i.e. $\left\{\begin{matrix} \frac{dx_1}{dt} = f_1(x_1,\cdots,x_n) \\ \vdots \\ \frac{dx_n}{dt} = f_n(x_1,\cdots,x_n)\end{matrix}\right\}$. We are interested in knowing how a system defined in this way evolves over time for any given initial condition. This describes what we call a **continuous dynamical system**. We call the set of all such solution curves the **flow** of the dynamical system.

If you imagine a vector field as describing a flowing liquid, then these parameterized curves simply describe what happens if you drop a particle into the flow and see where it goes as it carried by the flow. This is a good way to think about a continuous dynamical system even when the variables are describing such things as populations or economic variables rather than geometric coordinates. We'll still refer to the solutions as the flow of the system even though there's nothing physical about this flow.

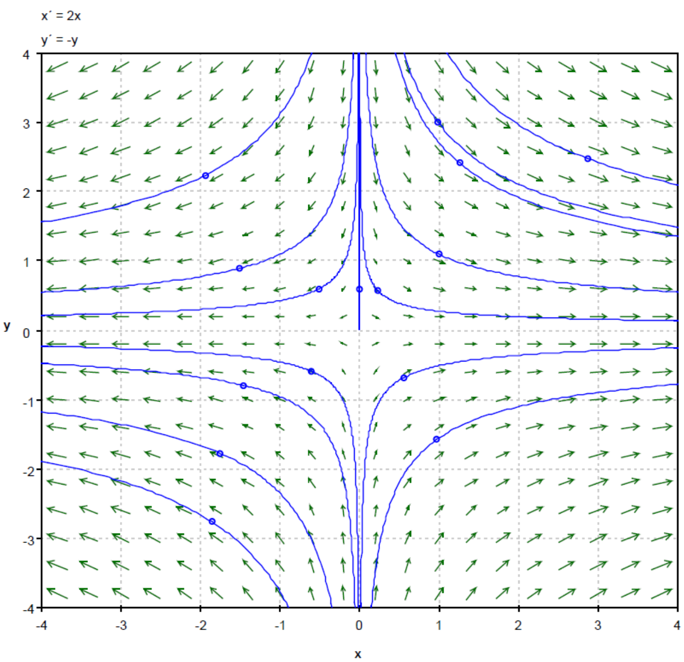
We are typically interested in the long-term behavior of such a system, but we often would also like to predict exactly where the particle will be after a specified time *t*, i.e. formulas for how the component functions evolve in time. In general, if the component functions of the underlying vector field are nonlinear, it's very difficult to find a tidy formula for how the system evolves over time. The linear case, on the other hand, is completely solvable using matrix methods.

Definition: A linear continuous dynamical system is a system of first-order differential equations of the form $\left\{\begin{matrix} \frac{dx_1}{dt} = a_{11}x_1+\cdots+a_{1n}x_n \\ \vdots \\ \frac{dx_n}{dt} = a_{n1}x_1+\cdots+a_{nn}x_n\end{matrix}\right\}$. If $\mathbf{x}=\begin{bmatrix} x_1 \\ \vdots \\ x_n\end{bmatrix}$, then $\frac{d\mathbf{x}}{dt}=\begin{bmatrix} \frac{dx_1}{dt} \\ \vdots \\ \frac{dx_n}{dt}\end{bmatrix}=\begin{bmatrix} a_{11}x_1+\cdots+a_{1n}x_n \\ \vdots \\ a_{n1}x_1+\cdots+a_{nn}x_n\end{bmatrix}=\begin{bmatrix} a_{11} & \cdots & a_{11} \\ \vdots & \ddots & \vdots \\ a_{11} & \cdots & a_{11}\end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_n\end{bmatrix}=\mathbf{A}\mathbf{x}$ where $\mathbf{A}=\begin{bmatrix} a_{11} & \cdots & a_{11} \\ \vdots & \ddots & \vdots \\ a_{11} & \cdots & a_{11}\end{bmatrix}$ is the matrix of coefficients. That is, $\frac{d\mathbf{x}}{dt}=\mathbf{A}\mathbf{x}$ where **A** is an $n\times n$ real matrix.

Situation: You want to solve a system of first-order linear differential equations of the form $\frac{d\mathbf{x}}{dt}=\mathbf{A}\mathbf{x}$ given some initial condition $\mathbf{x}(0)=\mathbf{x}_0$. How is this most efficiently accomplished?

Example 1: The simplest linear continuous dynamical system is the single equation $\frac{dx}{dt} = kx$ with initial condition $x(0) = x_0$. This is something we solved in basic calculus and yields exponential growth or decay (depending on whether $k > 0$ or $k < 0$). Specifically, we write $\frac{1}{x}\frac{dx}{dt} = k$ and integrate both sides to get $\ln|x(t)| = kt + c$ for some arbitrary constant c. [Many people choose to do this calculation as $\frac{dx}{x} = kdt$ and integrate both sides to get $\int \frac{dx}{x} = \int kdt \quad \Rightarrow \quad \ln|x| = kt + c$.] In any case, exponentiating both sides gives $|x(t)| = e^{kt+c} = e^c e^{kt} = ae^{kt}$, and we can remove the absolute value by allowing the constant a to be either positive or negative, so we get $x(t) = ae^{kt}$. Using the initial condition $x(0) = x_0$ we see that $x(0) = a = x_0$, so the solution is $\boxed{x(t) = x_0 e^{kt}}$.

Uncoupled systems: We call a system uncoupled (or unlinked) if the rates of change of each of the variables do not depend on any of the other variables. In the linear case, this would mean a system of the form $\left\{\begin{matrix} \frac{dx_1}{dt} = k_1 x_1 \\ \vdots \\ \frac{dx_n}{dt} = k_n x_n \end{matrix}\right\}$ with initial conditions $x_1(0),\ldots,x_n(0)$. Note that such a system can be expressed in matrix form as $\frac{d\mathbf{x}}{dt} = \mathbf{Dx}$ where $\mathbf{D}$ is the diagonal matrix $\mathbf{D} = \begin{bmatrix} k_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & k_n \end{bmatrix}$. Solving this system is nothing more than solving the previous problem repeatedly with different rate constants and corresponding initial conditions. We get the solution



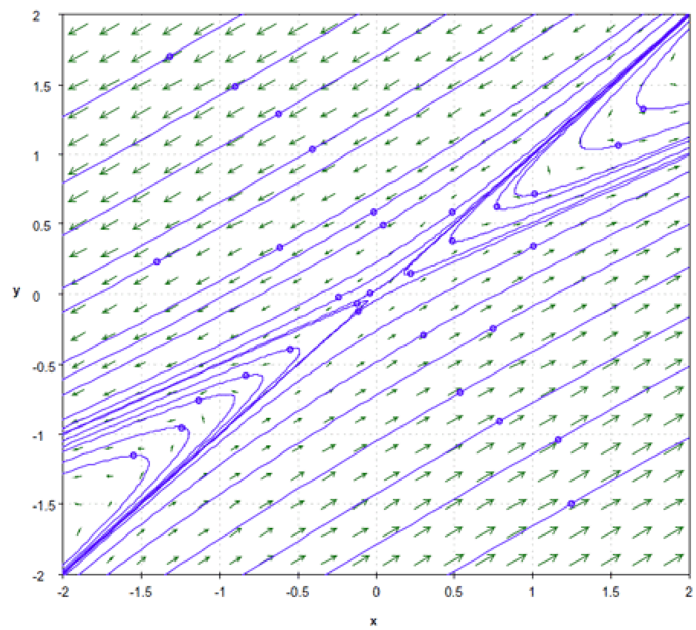
$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} = \begin{bmatrix} x_1(0)e^{k_1 t} \\ \vdots \\ x_n(0)e^{k_n t} \end{bmatrix} = \begin{bmatrix} e^{k_1 t} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{k_n t} \end{bmatrix} \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}.$$

Note that when $t = 0$ this matrix is just the identity matrix which simply reflects the fact that $t = 0$ corresponds to the initial conditions $\mathbf{x}(0) = \mathbf{x}_0$. Of greater interest is the fact that this time-varying matrix evolves over time to produce the flow emanating from any given initial condition. It is for this reason that we refer to this matrix as the **evolution matrix** for this uncoupled system. If we refer to this matrix as $[e^{t\mathbf{D}}]$, a notation that is perhaps best not taken too literally, then the system $\frac{d\mathbf{x}}{dt} = \mathbf{Dx}$ with initial conditions $\mathbf{x}(0) = \mathbf{x}_0$ has solution $\mathbf{x}(t) = [e^{t\mathbf{D}}]\mathbf{x}(0)$.

A **coupled system**, i.e. a system of the form $\frac{d\mathbf{x}}{dt} = \mathbf{Ax}$ where the matrix A is not diagonal, can often be solved by changing coordinates so that relative to some new basis (of eigenvectors) the system has a diagonal matrix. The tool at the heart of these methods is <u>diagonalization</u> or, in the case where a matrix cannot be diagonalized, finding an appropriate change of basis relative to which the underlying linear transformation has the simplest possible matrix representation (Jordan Canonical Form). The introduction of corresponding "evolution matrices" is a useful formalism for handling these general cases.

Problem: Solve the system $\begin{Bmatrix} \dfrac{dx}{dt} = 5x - 6y \\ \dfrac{dy}{dt} = 3x - 4y \end{Bmatrix}$ with initial conditions $x(0) = 3,\ y(0) = 1$.

Solution: In matrix form, we have $\dfrac{d\mathbf{x}}{dt} = \mathbf{Ax}$ where $\mathbf{A} = \begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix}$ and $\mathbf{x}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. If we use the PPLANE tool to get a sense of the underlying vector field, there are several things we might observe, but perhaps the most prominent features are the two lines passing through the origin one of which indicates growth (unstable) and the other decay (stable). Both of these are solution curves (trajectories) for this dynamical system, and the other trajectories appear to be some kind of superposition of these two prominent trajectories.



We refer to these prominent lines as **invariant directions**. What characterizes each of them is the fact that the **velocity vector field is parallel to each respective line**.

Note that if $\mathbf{x}$ is a position vector (from the origin) for any point on either of these lines, then at such a point we'll have $\mathbf{v} = \dfrac{d\mathbf{x}}{dt} = \mathbf{Ax} \parallel \mathbf{x}$ (parallel). This means that $\mathbf{Ax} = \lambda\mathbf{x}$ for some scalar λ. We refer to such a vector $\mathbf{x}$ as a **characteristic vector** or ***eigenvector*** for the matrix A, and we call the corresponding scalar λ its **characteristic value** or ***eigenvalue***. As we'll see, if we can use these eigenvectors as a basis for a better coordinate system, this will be the key to ***uncoupling*** such a system. It's easy to see that if $\mathbf{x}$ is any eigenvector then any scalar multiple of $\mathbf{x}$ is also an eigenvector, so the span of any such vector is an invariant direction or "***eigenspace***."

How do we find these eigenvalues and eigenvectors? We start by writing $\mathbf{Ax} = \lambda\mathbf{x} = \lambda\mathbf{Ix}$ where $\mathbf{I}$ is the identity matrix corresponding to the system (in this case a 2×2 identity matrix). We can then use some basic matrix algebra to write this as either $\mathbf{Ax} - \lambda\mathbf{Ix} = (\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ or $\lambda\mathbf{Ix} - \mathbf{Ax} = (\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$. So we are seeking a nonzero vector $\mathbf{x}$ that's in the kernel of the matrix $\lambda\mathbf{I} - \mathbf{A}$. The only way this can happen is if the matrix $\lambda\mathbf{I} - \mathbf{A}$ is ***not*** invertible (because if it was invertible the only solution would be the zero vector). We know that a simple criterion for invertibility of a square matrix is that its determine be nonzero, so to be *not* invertible means that *its determinant must be zero*. We refer to this as the **characteristic polynomial** of the matrix **A**, i.e. $p_{\mathbf{A}}(\lambda) = \det(\lambda\mathbf{I} - \mathbf{A})$. Its roots are the eigenvalues, and once we find them we can find the eigenvectors.

In this example, we start by finding the eigenvalues of the matrix: $\lambda\mathbf{I} - \mathbf{A} = \begin{bmatrix} \lambda - 5 & 6 \\ -3 & \lambda + 4 \end{bmatrix}$. Its characteristic polynomial is $p_{\mathbf{A}}(\lambda) = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1)$. This gives the eigenvalues $\lambda_1 = 2$ and $\lambda_2 = -1$. Using the first of these gives, we seek a vector $\mathbf{x} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ such that $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \begin{bmatrix} 2-5 & 6 \\ -3 & 2+4 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} -3 & 6 \\ -3 & 6 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. This gives redundant equations, but the requirement is that $-3\alpha + 6\beta = 0$ or, more simply, $\alpha = 2\beta$. Since any multiple of an eigenvector is also an eigenvector, this gives us some flexibility, but simplest is better. If we arbitrarily take $\beta = 1$, then $\alpha = 2$ and we get the eigenvector $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Similarly, the second eigenvalue gives the eigenvector $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. So we have $\left\{ \begin{matrix} \mathbf{A}\mathbf{v}_1 = \lambda_1 \mathbf{v}_1 = 2\mathbf{v}_1 \\ \mathbf{A}\mathbf{v}_2 = \lambda_2 \mathbf{v}_2 = -\mathbf{v}_2 \end{matrix} \right\}$. The first corresponds to a growth direction and the second to a decay direction.

These two vectors can be used to form the basis of a skew coordinate system relative to which the system will be uncoupled. Any position vector can be uniquely expressed as $\mathbf{x} = u_1\mathbf{v}_1 + u_2\mathbf{v}_2$, and therefore $\frac{d\mathbf{x}}{dt} = \frac{du_1}{dt}\mathbf{v}_1 + \frac{du_2}{dt}\mathbf{v}_2 = \mathbf{A}(u_1\mathbf{v}_1 + u_2\mathbf{v}_2) = u_1\mathbf{A}\mathbf{v}_1 + u_2\mathbf{A}\mathbf{v}_2 = u_1\lambda_1\mathbf{v}_1 + u_2\lambda_2\mathbf{v}_2$. Looking at the individual components we see that $\frac{du_1}{dt} = \lambda_1 u_1$ and $\frac{du_2}{dt} = \lambda_2 u_2$ and we can then use what we know about uncouple systems to conclude that $u_1(t) = e^{\lambda_1 t}u_1(0)$ and $u_2(t) = e^{\lambda_2 t}u_2(0)$, so $\mathbf{x}(t) = u_1(t)\mathbf{v}_1 + u_2(t)\mathbf{v}_2 = u_1(0)e^{\lambda_1 t}\mathbf{v}_1 + u_2(0)e^{\lambda_2 t}\mathbf{v}_2$.

In our specific example with $\mathbf{x}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ we can write

$\mathbf{x}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = u_1(0)\mathbf{v}_1 + u_2(0)\mathbf{v}_2 = c_1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \mathbf{S}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, so $\mathbf{S}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \mathbf{x}(0)$ and $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \mathbf{S}^{-1}\mathbf{x}(0) = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} u_1(0) \\ u_2(0) \end{bmatrix}$.

So $\mathbf{x}(t) = 2e^{2t}\mathbf{v}_1 - e^{-t}\mathbf{v}_2 = 2e^{2t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} - e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4e^{2t} - e^{-t} \\ 2e^{2t} - e^{-t} \end{bmatrix} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$.

Solving systems using diagonalization and evolution matrices

Given an $n \times n$ matrix **A**, suppose **S** is a change of basis matrix corresponding to either diagonalization or reduction to Jordan Canonical Form. We will have $\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{B}$ in this case, where **B** is diagonal or otherwise in simplest form. We then calculate $\mathbf{A} = \mathbf{S}\mathbf{B}\mathbf{S}^{-1}$, and substitution gives $\frac{d\mathbf{x}}{dt} = \mathbf{S}\mathbf{B}\mathbf{S}^{-1}\mathbf{x}$.

Multiplying on the left by $\mathbf{S}^{-1}$ and using the basic calculus fact that $\frac{d}{dt}(\mathbf{M}\mathbf{x}) = \mathbf{M}\frac{d\mathbf{x}}{dt}$ for any (constant) matrix **M**, we have $\mathbf{S}^{-1}\frac{d\mathbf{x}}{dt} = \frac{d(\mathbf{S}^{-1}\mathbf{x})}{dt} = \mathbf{B}(\mathbf{S}^{-1}\mathbf{x})$. If we write $\mathbf{u} = \mathbf{S}^{-1}\mathbf{x} = [\mathbf{x}]_{\mathscr{B}}$, where $\mathscr{B}$ is the new, preferred basis, then in these new coordinates the system becomes $\frac{d\mathbf{u}}{dt} = \mathbf{B}\mathbf{u}$, but now the system will be much more straightforward to solve.

The diagonalizable case

In the case where **B** is a diagonal matrix with the eigenvalues of **A** on the diagonal, the system is just

$$\frac{d\mathbf{u}}{dt} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}\mathbf{u} \text{ or } \left\{ \begin{matrix} \frac{du_1}{dt} = \lambda_1 u_1 \\ \vdots \\ \frac{du_n}{dt} = \lambda_n u_n \end{matrix} \right\}.$$

This has the solution $\left\{ \begin{matrix} u_1(t) = e^{\lambda_1 t}u_1(0) \\ \vdots \\ u_n(t) = e^{\lambda_n t}u_n(0) \end{matrix} \right\}$ or $\mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix}\begin{bmatrix} u_1(0) \\ \vdots \\ u_n(0) \end{bmatrix} = [e^{t\mathbf{B}}]\mathbf{u}(0)$.

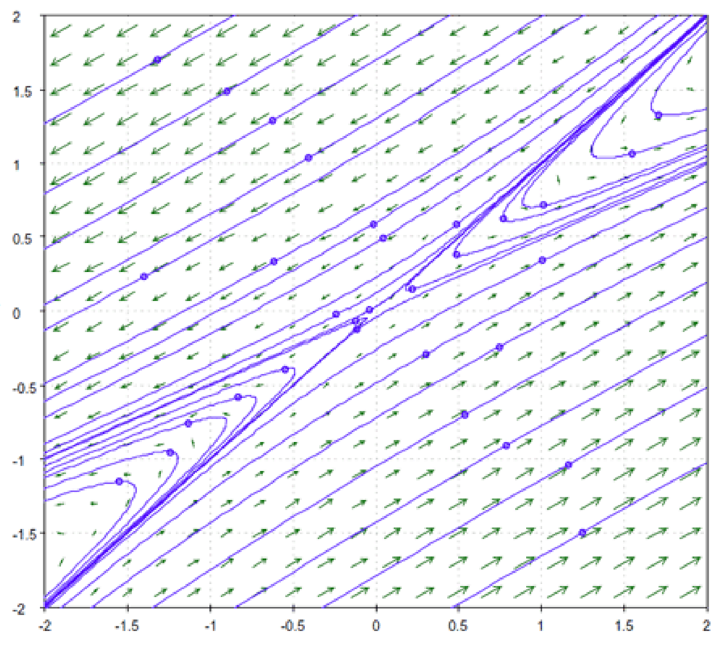
To revert back to the original coordinates, we write $\mathbf{x}=\mathbf{S}\mathbf{u}$, so $\mathbf{x}(t)=\mathbf{S}\mathbf{u}(t)=\mathbf{S}[e^{t\mathbf{B}}]\mathbf{u}(0)=\mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1}\mathbf{x}(0)$. If we denote the evolution matrix for the system in its original coordinates as $[e^{t\mathbf{A}}]$ where $\mathbf{x}(t)=[e^{t\mathbf{A}}]\mathbf{x}(0)$, then the previous calculation gives the simple relation $[e^{t\mathbf{A}}]=\mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1}$.

In other words, the evolution matrices for the solution are in the same relationship as the matrices **A** and **B**, namely $\mathbf{A}=\mathbf{SBS}^{-1}$. This pattern is very easy to remember, and this same pattern will again be the case where **B** is not diagonal but where the corresponding evolution matrix is still relatively easy to calculate.

$$\mathbf{A}=\mathbf{SBS}^{-1} \quad\Rightarrow\quad [e^{t\mathbf{A}}]=\mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1}, \text{ and the solution of the original system will be } \boxed{\mathbf{x}(t)=[e^{t\mathbf{A}}]\mathbf{x}(0)}.$$

The Main Idea:

Given a system of 1st order linear differential equations $\frac{d\mathbf{x}}{dt}=\mathbf{A}\mathbf{x}$ with initial conditions $\mathbf{x}(0)$, we use eigenvalue-eigenvector analysis to find an appropriate basis $\mathscr{B}=\{\mathbf{v}_1,\cdots,\mathbf{v}_n\}$ for $\mathbf{R}^n$ and a change of basis matrix $\mathbf{S}=\begin{bmatrix}\uparrow & & \uparrow\\ \mathbf{v}_1 & \cdots & \mathbf{v}_n\\ \downarrow & & \downarrow\end{bmatrix}$ such that in coordinates relative to this basis ($\mathbf{u}=\mathbf{S}^{-1}\mathbf{x}$) the system is in a standard form with a known solution. Specifically, we find a standard matrix $\mathbf{B}=[\mathbf{A}]_{\mathscr{B}}=\mathbf{S}^{-1}\mathbf{AS}$, transform the system into $\frac{d\mathbf{u}}{dt}=\mathbf{B}\mathbf{u}$, solve it as $\mathbf{u}(t)=[e^{t\mathbf{B}}]\mathbf{u}(0)$ where $[e^{t\mathbf{B}}]$ is the ***evolution matrix*** for **B**, then transform back to the original coordinates to get $\mathbf{x}(t)=[e^{t\mathbf{A}}]\mathbf{x}(0)$ where $[e^{t\mathbf{A}}]=\mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1}$ is the ***evolution matrix*** for **B**. That is $\mathbf{x}(t)=[e^{t\mathbf{A}}]=\mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1}\mathbf{x}(0)$. This is easier to do than it is to explain, so here are a few illustrative examples:



The diagonalizable case

Problem: Solve the system $\left\{\begin{aligned}\frac{dx}{dt}&=5x-6y\\ \frac{dy}{dt}&=3x-4y\end{aligned}\right\}$ with initial conditions $x(0)=3,\ y(0)=1$.

Solution: In matrix form, we have $\frac{d\mathbf{x}}{dt}=\mathbf{A}\mathbf{x}$ where $\mathbf{A}=\begin{bmatrix}5 & -6\\ 3 & -4\end{bmatrix}$ and $\mathbf{x}(0)=\begin{bmatrix}3\\1\end{bmatrix}$. We start by finding the eigenvalues of the matrix: $\lambda\mathbf{I}-\mathbf{A}=\begin{bmatrix}\lambda-5 & 6\\ -3 & \lambda+4\end{bmatrix}$, and the characteristic polynomial is $p_{\mathbf{A}}(\lambda)=\lambda^2-\lambda-2=(\lambda-2)(\lambda+1)$. This gives the eigenvalues $\lambda_1=2$ and $\lambda_2=-1$. The first of these gives the eigenvector $\mathbf{v}_1=\begin{bmatrix}2\\1\end{bmatrix}$, and the second gives the eigenvector $\mathbf{v}_2=\begin{bmatrix}1\\1\end{bmatrix}$. So we have $\left\{\begin{aligned}\mathbf{A}\mathbf{v}_1&=\lambda_1\mathbf{v}_1\\ \mathbf{A}\mathbf{v}_2&=\lambda_2\mathbf{v}_2\end{aligned}\right\}$. The change of basis matrix is $\mathbf{S}=\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}$ and with the new basis (of eigenvectors) $\mathscr{B}=\{\mathbf{v}_1,\mathbf{v}_2\}$ we have $[\mathbf{A}]_{\mathscr{B}}=\mathbf{S}^{-1}\mathbf{AS}=\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}=\begin{bmatrix}2 & 0\\ 0 & -1\end{bmatrix}=\mathbf{D}$, a diagonal matrix. [There is no need to carry out the multiplication of the matrices if $\mathcal{B}=\{\mathbf{v}_1,\mathbf{v}_2\}$ is known to be is a basis of eigenvectors. It will always yield a diagonal matrix with the eigenvalues on the diagonal.]

The evolution matrix for this diagonal matrix is $[e^{t\mathbf{D}}]=\begin{bmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{bmatrix}$, and the solution of the system is:

$$\mathbf{x}(t)=[e^{t\mathbf{A}}]\mathbf{x}(0)=\mathbf{S}[e^{t\mathbf{D}}]\mathbf{S}^{-1}\mathbf{x}(0)=\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix}=\begin{bmatrix} 2e^{2t} & e^{-t} \\ e^{2t} & e^{-t} \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$=\begin{bmatrix} 4e^{2t}-e^{-t} \\ 2e^{2t}-e^{-t} \end{bmatrix}=2e^{2t}\begin{bmatrix} 2 \\ 1 \end{bmatrix}-e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}=2e^{2t}\mathbf{v}_1-e^{-t}\mathbf{v}_2$$

Moral of the Story: It's always possible to find a special basis relative to which a given linear system is in its simplest possible form. The new basis provides a way to decompose the given problem into several simple, standard problems which can be easily solved. Any complication in the algebraic expressions for the solution is the result of changing back to the original coordinates.

The standard 2×2 cases are:

Diagonalizable with eigenvalues λ_1,λ_2: $\quad \mathbf{B}=\mathbf{D}=\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad [e^{t\mathbf{B}}]=[e^{t\mathbf{D}}]=\begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$

Complex pair of eigenvalues $\lambda=a\pm ib$: $\quad \mathbf{B}=\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \quad [e^{t\mathbf{B}}]=e^{at}\begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix}$

Repeated eigenvalue λ **with** $\mathrm{GM}<\mathrm{AM}$: $\quad \mathbf{B}=\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \quad [e^{t\mathbf{B}}]=\begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix}=e^{\lambda t}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$

In general, you should expect to encounter systems more complicated than these 2×2 examples.