Coordinates Relative to a Basis; Matrix of a Linear Transformation Relative to Bases

Coordinates relative to a basis

Perhaps the single most important thing about having a basis for a subspace is that there is only one way to express any vector in the subspace in terms of the given basis. This brings us to the definition of coordinates. But first, we have to prove the following proposition:

Proposition: Suppose $V \subseteq \mathbf{R}^n$ is a subspace (which could be all of $\mathbf{R}^n$ or any proper subspace) and let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a basis for V (hence $\dim(V) = k$). Then any vector $\mathbf{x} \in V$ can be uniquely expressed as $\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k$ for some scalars $\{c_1, \cdots, c_k\}$. The scalars are called the **coordinates of x relative to the basis** $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$.

In terms of matrices, we have $\mathbf{x} = \begin{bmatrix} \uparrow & & \uparrow \\ \mathbf{v}_1 & \cdots & \mathbf{v}_k \\ \downarrow & & \downarrow \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix} = \mathbf{S}[\mathbf{x}]_{\mathcal{B}}$. This says simply that the system of linear equations $\mathbf{S}[\mathbf{x}]_{\mathcal{B}} = \mathbf{x}$ must yield a unique solution $[\mathbf{x}]_{\mathcal{B}}$, and we refer to this as the **coordinate vector** for **x** relative to the basis $\mathcal{B}$.

Proof: Suppose there were two different ways to express a vector $\mathbf{x} \in V$ in terms of the basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$. We could then write $\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k = d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \ldots + d_k\mathbf{v}_k$. But we can then transpose to rewrite this as $(c_1 - d_1)\mathbf{v}_1 + (c_2 - d_2)\mathbf{v}_2 + \ldots + (c_k - d_k)\mathbf{v}_k = \mathbf{0}$. Because $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a basis, these vectors must be linearly independent. Therefore $(c_1 - d_1) = (c_2 - d_2) = \ldots = (c_k - d_k) = 0$, so $c_1 = d_1, c_2 = d_2, \ldots, c_k = d_k$. That is, there is only one way to express any vector in terms of a basis.

Example 1: The plane in $\mathbf{R}^3$ passing through the origin with normal vector $\mathbf{n} = \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}$ is a subspace with $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\} = \left\{ \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ as a basis. (There are infinitely many such choices for a basis. All you have to do is choose two nonparallel vectors in the plane, each of which must be perpendicular to **n**. The dot product of each of them with **n** must be 0.) You can easily verify that the vector $\mathbf{x} = \begin{bmatrix} 1 \\ -7 \\ 2 \end{bmatrix}$ is in this subspace. What are its coordinates relative to the basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$, i.e. what is $[\mathbf{x}]_{\mathcal{B}}$?

Solution: Our definition above tells us that $\mathbf{S}[\mathbf{x}]_{\mathcal{B}} = \mathbf{x}$ where $\mathbf{S} = \begin{bmatrix} 2 & 1 \\ -2 & 1 \\ 1 & 0 \end{bmatrix}$, so we solve this inhomogeneous system as $\left[\begin{array}{cc|c} 2 & 1 & 1 \\ -2 & 1 & -7 \\ 1 & 0 & 2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{array}\right]$. Therefore $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$, i.e. $\mathbf{x} = 2\mathbf{v}_1 - 3\mathbf{v}_2$, and you can easily verify that this is the case. It's worth noting that had we chosen $\mathbf{x} \notin V$, the system would have been inconsistent.

The method described above for finding the coordinates of a vector relative to a given basis is completely general and should always be used in the case of proper subspaces (not the whole space). However, in the special case where the subspace is the whole space, i.e. $V = \mathbf{R}^n$ with basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, we have another method available for finding coordinates and for relating the coordinates.

Special case where $V = \mathbf{R}^n$ with basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$: In this case, the "change of basis matrix" **S** is an $n \times n$ matrix $\mathbf{S} = \begin{bmatrix} \uparrow & & \uparrow \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ \downarrow & & \downarrow \end{bmatrix}$. It's columns are linearly independent and its rank is *n*, so it's *invertible*. [Note that this would not have been meaningful for a subspace with dimension $k < n$.]

So we have $\mathbf{x} = \mathbf{S}[\mathbf{x}]_{\mathcal{B}}$ and $[\mathbf{x}]_{\mathcal{B}} = \mathbf{S}^{-1}\mathbf{x}$.

Example 2: Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$. You can easily show that these three vectors are linearly independent and therefore form a basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ for $\mathbf{R}^3$. Find the coordinates of the vector $\mathbf{x} = \begin{bmatrix} 7 \\ 1 \\ 2 \end{bmatrix}$ relative to this basis.

Solution: You can solve this in (at least) two ways. First, we could use the general method for finding the coordinates of a vector relative to a basis: $\left[\begin{array}{ccc|c} 1 & 3 & 2 & 7 \\ 1 & 2 & 0 & 1 \\ 2 & -1 & 1 & 2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1/11 \\ 0 & 1 & 0 & 6/11 \\ 0 & 0 & 1 & 30/11 \end{array}\right]$, so $\mathbf{x} = -\frac{1}{11}\mathbf{v}_1 + \frac{6}{11}\mathbf{v}_2 + \frac{30}{11}\mathbf{v}_3$.

You can plug in the components and check this, if you like.

Alternatively, let $\mathbf{S} = \begin{bmatrix} 1 & 3 & 2 \\ 1 & 2 & 0 \\ 2 & -1 & 1 \end{bmatrix}$ be the change of basis matrix. You can calculate its inverse by hand or by calculator to get $\mathbf{S}^{-1} = \frac{1}{11}\begin{bmatrix} -2 & 5 & 4 \\ 1 & 3 & -2 \\ 5 & -7 & 1 \end{bmatrix}$. So $[\mathbf{x}]_{\mathcal{B}} = \mathbf{S}^{-1}\mathbf{x} = \frac{1}{11}\begin{bmatrix} -2 & 5 & 4 \\ 1 & 3 & -2 \\ 5 & -7 & 1 \end{bmatrix}\begin{bmatrix} 7 \\ 1 \\ 2 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} -1 \\ 6 \\ 30 \end{bmatrix} = \begin{bmatrix} -1/11 \\ 6/11 \\ 30/11 \end{bmatrix}$.

Matrix of a linear transformation relative to an alternate basis

The fact that we can speak of the coordinates of a vector relative to a basis other than the standard basis allows us to think of the matrix of a linear transformation in a much richer (though possibly a little more abstract) way. Though we could develop this perspective more generally for any linear transformation $T: \mathbf{R}^n \to \mathbf{R}^m$, we'll specialize to the case where **A** is an $n \times n$ (square) matrix representing a linear transformation $T: \mathbf{R}^n \to \mathbf{R}^n$ by $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$. Up to this point, we have only had standard coordinates and the standard basis $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ and our understanding of the entries of a matrix was greatly restricted by this. Specifically, relative to the basis $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ a matrix $\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$ had the interpretation that:

$$\mathbf{A}\mathbf{e}_1 = \begin{bmatrix} a_{11} \\ \vdots \\ a_{n1} \end{bmatrix} = a_{11}\mathbf{e}_1 + \cdots a_{n1}\mathbf{e}_n, \quad \mathbf{A}\mathbf{e}_2 = \begin{bmatrix} a_{12} \\ \vdots \\ a_{n2} \end{bmatrix} = a_{12}\mathbf{e}_1 + \cdots a_{n2}\mathbf{e}_n, \quad \ldots \quad , \mathbf{A}\mathbf{e}_n = \begin{bmatrix} a_{1n} \\ \vdots \\ a_{nn} \end{bmatrix} = a_{1n}\mathbf{e}_1 + \cdots a_{nn}\mathbf{e}_n.$$

In other words, the entries in each column tell us the coordinates of the image of each standard basis vector relative to the standard basis. We can write this as $\mathbf{A}=\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}=\begin{bmatrix} \uparrow & & \uparrow \\ [\mathbf{Ae}_1]_{\mathcal{E}} & \cdots & [\mathbf{Ae}_n]_{\mathcal{E}} \\ \downarrow & & \downarrow \end{bmatrix}$.

Expanding our viewpoint a bit, if we can do this all relative to the standard basis, why not do the same thing relative to an alternate basis $\mathcal{B}=\{\mathbf{v}_1,\mathbf{v}_2,\dots,\mathbf{v}_n\}$ for $\mathbf{R}^n$?

Definition: The matrix of a linear transformation $T:\mathbf{R}^n\to\mathbf{R}^n$ relative to the basis $\mathcal{B}=\{\mathbf{v}_1,\mathbf{v}_2,\dots,\mathbf{v}_n\}$ is the matrix $[T]_{\mathcal{B}}=\begin{bmatrix} \uparrow & & \uparrow \\ [T(\mathbf{v}_1)]_{\mathcal{B}} & \cdots & [T(\mathbf{v}_n)]_{\mathcal{B}} \\ \downarrow & & \downarrow \end{bmatrix}$.

If the linear transformation T is represented by the matrix $\mathbf{A}$ relative to the standard basis $\mathcal{E}=\{\mathbf{e}_1,\mathbf{e}_2,\dots,\mathbf{e}_n\}$, we often simply write $[\mathbf{A}]_{\mathcal{B}}=\begin{bmatrix} \uparrow & & \uparrow \\ [A\mathbf{v}_1]_{\mathcal{B}} & \cdots & [A\mathbf{v}_n]_{\mathcal{B}} \\ \downarrow & & \downarrow \end{bmatrix}$.

Example 3: Let's take another look at a previous example where we had $\{\mathbf{v}_1,\mathbf{v}_2\}=\left\{\begin{bmatrix}2\\-2\\1\end{bmatrix},\begin{bmatrix}1\\1\\0\end{bmatrix}\right\}$ spanning a plane with normal vector $\mathbf{n}=\begin{bmatrix}-1\\1\\4\end{bmatrix}$. Let's let $\mathbf{v}_3=\mathbf{n}$ and include this with the other two vectors to form a basis $\mathcal{B}=\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}=\left\{\begin{bmatrix}2\\-2\\1\end{bmatrix},\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}-1\\1\\4\end{bmatrix}\right\}$ for all of $\mathbf{R}^3$. Now consider the linear transformation that reflects any vector $\mathbf{x}$ in $\mathbf{R}^3$ through the plane spanned by the first two vectors. What is the matrix of this linear transformation relative to the basis $\mathcal{B}=\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$?

Solution: This is actually extremely easy, perhaps we should even say obvious. Observe that:

$$\begin{matrix} \mathbf{v}_1 \xrightarrow{T} \mathbf{v}_1 = 1\mathbf{v}_1+0\mathbf{v}_2+0\mathbf{v}_3 \\ \mathbf{v}_2 \xrightarrow{T} \mathbf{v}_2 = 0\mathbf{v}_1+1\mathbf{v}_2+0\mathbf{v}_3 \\ \mathbf{v}_3 \xrightarrow{T} -\mathbf{v}_3 = 0\mathbf{v}_1+0\mathbf{v}_2-1\mathbf{v}_3 \end{matrix} \Rightarrow [T]_{\mathcal{B}}=\begin{bmatrix} \uparrow & \uparrow & \uparrow \\ [T(\mathbf{v}_1)]_{\mathcal{B}} & [T(\mathbf{v}_2)]_{\mathcal{B}} & [T(\mathbf{v}_3)]_{\mathcal{B}} \\ \downarrow & \downarrow & \downarrow \end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix}=[\mathbf{A}]_{\mathcal{B}}$$

The simplicity of this transformation is reflected (pardon the pun) by the simplicity of its matrix relation to a well-chosen basis.

Relating matrices of a linear transformation relative to different bases

Now that we've opened the door to the possibility of representing a linear transformation by different matrices corresponding to different bases, it's important to know how to relate these matrices. Though we can reason through this algebraically, there's a much more elegant approach that uses what is known as a ***commutative diagram***. In this case, think of a linear transformation as some kind of action and think of the choice of basis as analogous to the choice of a language. For example, let's say we choose to think of the standard basis $\mathcal{E}=\{\mathbf{e}_1,\mathbf{e}_2,\dots,\mathbf{e}_n\}$ as English and an alternative basis $\mathcal{B}=\{\mathbf{v}_1,\mathbf{v}_2,\dots,\mathbf{v}_n\}$ as Bulgarian.

In English, we might denote the action of the linear transformation as $\{\mathbf{R}^n,\mathcal{E}\}\xrightarrow{\mathbf{A}}\{\mathbf{R}^n,\mathcal{E}\}$.

In Bulgarian, we might denote the action of the linear transformation as $\left\{\mathbf{R}^n, \mathscr{B}\right\} \xrightarrow{[\mathbf{A}]_{\mathscr{B}}} \left\{\mathbf{R}^n, \mathscr{B}\right\}$.

Note that the domain and codomain has been appended in each case by the chosen "language." It's best to think of the transformation as going from left to right in this formulation. Vectors in $\mathbf{R}^n$ are in each case expressed in coordinates relative to the specified basis.

How do we relate vectors (either in the domain or the codomain) from one language to the other? This is where the relations $\mathbf{x} = \mathbf{S}\left[\mathbf{x}\right]_{\mathscr{B}}$ and $\left[\mathbf{x}\right]_{\mathscr{B}} = \mathbf{S}^{-1}\mathbf{x}$ come in. If we think of changing languages as moving up and down in the diagram with English on the top line and Bulgarian on the bottom line, we have:

$$\begin{array}{ccc} \left\{\mathbf{R}^n, \mathcal{E}\right\} & \xrightarrow{\mathbf{A}} & \left\{\mathbf{R}^n, \mathcal{E}\right\} \\ \mathbf{S}\uparrow & & \mathbf{S}\uparrow \\ \left\{\mathbf{R}^n, \mathscr{B}\right\} & \xrightarrow{\left[\mathbf{A}\right]_{\mathscr{B}}} & \left\{\mathbf{R}^n, \mathscr{B}\right\} \end{array}$$

Note, in particular, which way the vertical arrows go based on the fact that $\mathbf{x} = \left[\mathbf{x}\right]_{\mathcal{E}} = \mathbf{S}\left[\mathbf{x}\right]_{\mathscr{B}}$.
If we start with a Bulgarian vector, we can either (a) translate it in to English and then apply the English version of the matrix, or (b) carry out the transformation in Bulgarian and then change the language to English. The results should be the same if the transformation has any objective meaning.

Algebraically this gives $\mathbf{AS} = \mathbf{S}\left[\mathbf{A}\right]_{\mathscr{B}}$, but we usually express this as either $\boxed{\left[\mathbf{A}\right]_{\mathscr{B}} = \mathbf{S}^{-1}\mathbf{AS}}$ or $\boxed{\mathbf{A} = \mathbf{S}\left[\mathbf{A}\right]_{\mathscr{B}}\mathbf{S}^{-1}}$.

Definition: Two $n \times n$ matrices **A** and **B** are called **similar** if there is an invertible $n \times n$ matrix **S** such that $\mathbf{B} = \mathbf{S}^{-1}\mathbf{AS}$.

Said differently, two matrices are similar if they represent the same linear transformation relative to two different bases.

There's another way to see this algebraic relationship without a diagrammatic roadmap. If we interpret the columns of the change of basis matrix **S** as well as the definition of $\left[\mathbf{A}\right]_{\mathscr{B}}$ we observe that:

$$\left\{\begin{matrix} \mathbf{Se}_1 = \mathbf{v}_1 \\ \vdots \\ \mathbf{Se}_n = \mathbf{v}_n \end{matrix}\right\} \Rightarrow \left\{\begin{matrix} \mathbf{ASe}_1 = \mathbf{Av}_1 \\ \vdots \\ \mathbf{ASe}_n = \mathbf{Av}_n \end{matrix}\right\} \Rightarrow \left\{\begin{matrix} \mathbf{S}^{-1}\mathbf{ASe}_1 = \mathbf{S}^{-1}\left(\mathbf{Av}_1\right) \\ \vdots \\ \mathbf{S}^{-1}\mathbf{ASe}_n = \mathbf{S}^{-1}\left(\mathbf{Av}_n\right) \end{matrix}\right\} \Rightarrow \left\{\begin{matrix} \left(\mathbf{S}^{-1}\mathbf{AS}\right)\mathbf{e}_1 = \left[\mathbf{Av}_1\right]_{\mathscr{B}} \\ \vdots \\ \left(\mathbf{S}^{-1}\mathbf{AS}\right)\mathbf{e}_n = \left[\mathbf{Av}_n\right]_{\mathscr{B}} \end{matrix}\right\} \Rightarrow \mathbf{S}^{-1}\mathbf{AS} = \left[\mathbf{A}\right]_{\mathscr{B}}$$

Example 4: Suppose we have the basis $\mathscr{B} = \left\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\right\} = \left\{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right\}$ for $\mathbf{R}^3$ and that a linear transformation is defined in terms of how it acts on these basis vectors with $\left\{\begin{matrix} T(\mathbf{v}_1) = 2\mathbf{v}_2 \\ T(\mathbf{v}_2) = -\mathbf{v}_3 \\ T(\mathbf{v}_3) = \mathbf{v}_1 + \mathbf{v}_2 \end{matrix}\right\}$. If we denote the matrix of this linear transformation relative to the standard basis by **A** and relative to the basis $\mathscr{B}$ by $\left[\mathbf{A}\right]_{\mathscr{B}}$, find the matrix **A**.

Solution: There is no calculation necessary in determining the matrix $\left[\mathbf{A}\right]_{\mathscr{B}}$. This is actually extremely easy, perhaps we should even say obvious. Observe that:

$$\left\{\begin{matrix} T(\mathbf{v}_1) = 2\mathbf{v}_2 \\ T(\mathbf{v}_2) = -\mathbf{v}_3 \\ T(\mathbf{v}_3) = \mathbf{v}_1 + \mathbf{v}_2 \end{matrix}\right\} \Rightarrow \left[T\right]_{\mathscr{B}} = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \left[T(\mathbf{v}_1)\right]_{\mathscr{B}} & \left[T(\mathbf{v}_2)\right]_{\mathscr{B}} & \left[T(\mathbf{v}_3)\right]_{\mathscr{B}} \\ \downarrow & \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} = \left[\mathbf{A}\right]_{\mathscr{B}}$$

The simplicity of how this linear transformation is defined relative to the basis $\mathcal{B}$ yields a correspondingly simple matrix $[\mathbf{A}]_{\mathcal{B}}$ relative to this basis. To determine the matrix **A** relative to the standard basis, we use $[\mathbf{A}]_{\mathcal{B}} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$ and solve for $\mathbf{A} = \mathbf{S}[\mathbf{A}]_{\mathcal{B}}\mathbf{S}^{-1}$. From the basis we have $\mathbf{S} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 2 \\ -1 & -1 & 3 \end{bmatrix}$ and we can calculate

$$\mathbf{S}^{-1} = \tfrac{1}{14}\begin{bmatrix} 8 & 2 & -4 \\ -5 & 4 & -1 \\ 1 & 2 & 3 \end{bmatrix}, \text{ so } \mathbf{A} = \tfrac{1}{14}\begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 2 \\ -1 & -1 & 3 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 2 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} 8 & 2 & -4 \\ -5 & 4 & -1 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} -11/14 & -4/7 & 9/14 \\ 45/14 & 3/7 & -5/14 \\ -3/14 & -10/7 & 5/14 \end{bmatrix}.$$

The moral is that if we choose to work only with the standard basis, and if a linear transformation does not act in a simple way relative to the standard basis, then its standard matrix will most likely not be very simple. Finding a basis relative to which a given linear transformation acts simply will be a central idea in the course in the coming weeks.

Example 5: The basis $\mathcal{B}$ in the above example is such that the 3rd basis vector is perpendicular (orthogonal) to the first two vectors. Suppose $\mathbf{V} = \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ is the plane spanned by the first two vectors. Find the matrix for orthogonal projection of any vector onto this plane.

Solution: Relative to the given basis, we observe that $\begin{Bmatrix} T(\mathbf{v}_1) = \mathbf{v}_1 \\ T(\mathbf{v}_2) = \mathbf{v}_2 \\ T(\mathbf{v}_3) = \mathbf{0} \end{Bmatrix}$. So $[\mathbf{A}]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. As in the previous case, we have $\mathbf{A} = \mathbf{S}[\mathbf{A}]_{\mathcal{B}}\mathbf{S}^{-1} = \tfrac{1}{14}\begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 2 \\ -1 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 8 & 2 & -4 \\ -5 & 4 & -1 \\ 1 & 2 & 3 \end{bmatrix} = \tfrac{1}{14}\begin{bmatrix} 13 & -2 & -3 \\ -2 & 10 & -6 \\ -3 & -6 & 5 \end{bmatrix}$. This matrix will have rank 2 because its image is 2-dimensional. Once again we see that the matrix is complicated relative to the standard basis but quite simple relative to a basis that is well-suited to the transformation.

Notes by Robert Winters