Math E-21c – Fall 2025 – Problem Set #5

[80 points total]

Problem 1. (20 pts) For the equation $\ddot{x}+2\dot{x}+cx=0$, c constant, if we seek exponential solutions $x=e^{rt}$ and examine the resulting characteristic polynomial:

a) Tell which values of *c* correspond to each of the three cases: two real roots, repeated real root, and complex roots.

b) For the case of two real roots, tell for which values of *c* both roots are negative, both roots are positive, or the roots have different signs.

c) Summarize the above information by drawing a *c*-axis, and marking the intervals on it corresponding to the different possibilities for the roots of the characteristic equation.

d) Finally, use this information to mark the interval on the *c*-axis for which the corresponding ODE is stable. (The stability criterion using roots is what you will need.)

e) Specifically, solve the initial value problem: $\ddot{x}+2\dot{x}-3x=0, \quad x(0)=1, \quad \dot{x}(0)=-1$.

f) Specifically, solve the initial value problem: $\ddot{x}+2\dot{x}+5x=0, \quad x(0)=1, \quad \dot{x}(0)=-1$.

g) Specifically, solve the initial value problem: $\ddot{x}+2\dot{x}+x=0, \quad x(0)=1, \quad \dot{x}(0)=-1$.

In problems 2-4, find the general solution to the given ODE and also the specific solution satisfying the given initial conditions.

Problem 2. (5 pts) $2\ddot{x}-3\dot{x}=0, \quad x(0)=3, \quad \dot{x}(0)=1$

Problem 3. (5 pts) $\ddot{x}-6\dot{x}+25x=0, \quad x(0)=3, \quad \dot{x}(0)=1$.

Problem 4: (10 pts) Find all solutions to the following homogeneous ODEs:

a) $\dfrac{d^5x}{dt^5}-4\dfrac{d^4x}{dt^4}+4\dfrac{d^3x}{dt^3}=0$

b) $\dfrac{d^3x}{dt^3}+6\dfrac{d^2x}{dt^2}+12\dfrac{dx}{dt}+8x=0$

Problem 5. (10 pts) Find a <u>particular solution</u> to the differential equation $\ddot{x}-2\dot{x}+4x=e^t\cos t$. Use complex exponentials where possible.

Problem 6. (10 pts) Find the <u>unique solution</u> to the differential equation $\ddot{x}-6\dot{x}+9x=4e^{3t}$ with initial conditions $x(0)=5$, $\dot{x}(0)=6$.

Problem 7: (10 pts) [See Lecture Notes #5 for references.] A resistor of 12 ohms is connected in series with an inductor of one henry, a capacitor of 0.01 farads, and a voltage source supplying $12\cos 10t$ volts. [This is a consistent system of units of measurement.] At $t=0$, the charge on the capacitor is zero and the current in the circuit is also zero.

a) Determine $Q(t)$, the charge on the capacitor as a function of time for $t>0$.

b) Determine $I(t)$, the current in the circuit as a function of time for $t>0$.

Problem 8. (10 pts) A driven mass-spring-dashpot system is modeled by the ODE: $m\ddot{x}+c\dot{x}+kx=F_0\cos\omega t$ with $m=1$, $c=6$, $k=45$, and $F_0=50$. Find the amplitude $A(\omega)$ of the response as a function of the input frequency ω and find the frequency which gives the largest system response.