Problem Set #14 – Math E-21c – Fall 2024

[Do these, but don't turn them in. Solutions will be posted. Nonlinear systems and analysis will be on the Final Exam.]

Problem 1. Consider the system: $\left\{\begin{aligned} \frac{dx}{dt} &= x(1-x+ky-k) \\ \frac{dy}{dt} &= y(1-y+kx-k) \end{aligned}\right\}$ where k is a constant different from 1 and –1.

a) The system above has exactly one equilibrium point (a,b) in the first quadrant with $a>0$ and $b>0$. Find this equilibrium point.

b) Find the Jacobian matrix at the equilibrium point.

c) Determine the stability of the equilibrium point. Your answer may depend on the constant k.

Problem 2: The interaction of two species of animals is modeled by

$$\left\{\begin{aligned} \frac{dx}{dt} &= x(2-x+y) \\ \frac{dy}{dt} &= y(4-x-y) \end{aligned}\right\} \text{ for } x \geq 0 \text{ and } y \geq 0.$$

a) Sketch a phase portrait for this system. Make sure that your sketch clearly shows the nullclines and the equilibria.

b) There is one equilibrium point (a,b) with $a>0$ and $b>0$. Find the Jacobian matrix **J** of the system at that point. [See Supplement for details on linearization. We'll discuss them in detail next week.]

c) Determine the stability of the equilibrium point (a,b) discussed in part (b) and provide a sketch of the solutions to the approximately linear system in the vicinity of the equilibrium. [**See Supplement.**]

Problem 3. Consider the system: $\left\{\begin{aligned} \frac{dx}{dt} &= x^2+y^2-1 \\ \frac{dy}{dt} &= xy \end{aligned}\right\}$.

Sketch a phase plane for this system. Make sure that your sketch clearly shows the nullclines and the equilibria. Which equilibria are stable? Use Jacobian analysis to analyze any equilibria.

Problem 4. The dynamics of a frictionless pendulum of length L are given by the system $\left\{\begin{aligned} \frac{d\alpha}{dt} &= \omega \\ \frac{d\omega}{dt} &= -\frac{g}{L}\sin\alpha \end{aligned}\right\}$

where α is the angle the rod of the pendulum makes with the vertical line, $\omega = \frac{d\alpha}{dt}$ is the angular velocity, and g is the gravitational constant.

a) Sketch a phase portrait for this system. Think about the trajectories in terms of the motion of a frictionless pendulum.

b) Find the Jacobian matrix at all equilibrium point, and compute the eigenvalues. What does the answer tell you about the stability of the equilibria?