Problem Set #13 – Math E-21c – Fall 2024
[60 points total]

Problem 1: (10 pts) Solve the system $\begin{Bmatrix} \frac{dx}{dt} = -7x + 9y \\ \frac{dy}{dt} = -4x + 5y \end{Bmatrix}$ with initial conditions $x(0) = 1$, $y(0) = -1$. Express your solution in terms of real-valued functions. Sketch the general flow of this system and, in particular the solution for the given initial conditions.

Problem 2: (20 pts)

a) For the system $\frac{d\mathbf{x}}{dt} = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} \mathbf{x} = \mathbf{B}\mathbf{x}$, find the evolution matrix $[e^{t\mathbf{B}}]$.

Refer to Problems 6, 7, and 8 from PS12 for ideas on how to solve this. Under what conditions will the zero state be a stable equilibrium?

b) The system $\begin{Bmatrix} \frac{dx}{dt} = -2x + y + 5z \\ \frac{dy}{dt} = -2x + 2y + 3z \\ \frac{dz}{dt} = -x + 3z \end{Bmatrix}$ with corresponding matrix $\mathbf{A} = \begin{bmatrix} -2 & 1 & 5 \\ -2 & 2 & 3 \\ -1 & 0 & 3 \end{bmatrix}$ has a repeated eigenvalue with algebraic multiplicity 3 but only one independent eigenvector. Find the eigenvalue λ and an eigenvector $\mathbf{v}_1$ as well as *generalized eigenvectors* $\mathbf{v}_2$ and $\mathbf{v}_3$ to form a basis $\mathscr{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ such that $\begin{Bmatrix} \mathbf{A}\mathbf{v}_1 = \lambda\mathbf{v}_1 \\ \mathbf{A}\mathbf{v}_2 = \mathbf{v}_1 + \lambda\mathbf{v}_2 \\ \mathbf{A}\mathbf{v}_3 = \mathbf{v}_2 + \lambda\mathbf{v}_3 \end{Bmatrix}$.

c) Use the basis in part (b) and the result of part (a) to find the unique solution to the system $\begin{Bmatrix} \frac{dx}{dt} = -2x + y + 5z \\ \frac{dy}{dt} = -2x + 2y + 3z \\ \frac{dz}{dt} = -x + 3z \end{Bmatrix}$ with initial conditions $x(0) = 1$, $y(0) = 1$, $z(0) = 2$.

Problem 3: (15 pts) Solve the system $\begin{Bmatrix} \frac{dx}{dt} = -x - 2y + 3 \\ \frac{dy}{dt} = 2x - y + 4 \end{Bmatrix}$ with initial conditions $x(0) = 1$, $y(0) = -1$. Express your solution in terms of real-valued functions. Sketch the general flow of this system and, in particular the solution for the given initial conditions. [*Hint*: Find the equilibrium and translate axes.]

Problem 4: (15 pts.) Find the <u>general solution</u> of the nonautonomous system $\begin{Bmatrix} \frac{dx}{dt} = x + y + e^{-2t} \\ \frac{dy}{dt} = 4x - 2y - 2e^{t} \end{Bmatrix}$ by relating it to a linear system. [***Reference***: Lectures Notes #13, pgs 4-9. Use either Undetermined Coefficients or Variation of Parameters to find particular solution.]