Problem Set #12 – Math E-21c – Fall 2024
[90 points total]

Problem 1: (10 pts) Consider a linear system $\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$ where **A** is an $n \times n$ matrix of arbitrary size.

(a) Suppose $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are solutions of this system. Is the sum $\mathbf{x}(t) = \mathbf{x}_1(t) + \mathbf{x}_2(t)$ a solution as well? Explain.

(b) Suppose $\mathbf{x}_1(t)$ is a solution of this system and k is an arbitrary constant. Is $\mathbf{x}(t) = k\,\mathbf{x}_1(t)$ a solution as well? Explain.

Problem 2: (10 pts) Solve the system $\left\{ \begin{aligned} \frac{dx}{dt} &= -x - 2y \\ \frac{dy}{dt} &= 2x - y \end{aligned} \right\}$ with initial conditions $x(0) = 1,\ y(0) = -1$. Express your solution in terms of <u>real-valued functions</u>. Sketch the general flow of this system and, in particular the solution for the given initial conditions.

Problem 3: (10 pts) Solve the system $\left\{ \begin{aligned} \frac{dx}{dt} &= -x + 5y \\ \frac{dy}{dt} &= -2x + 5y \end{aligned} \right\}$ with initial conditions $x(0) = 6,\ y(0) = 5$. Express your solution in terms of <u>real-valued functions</u>. Sketch the general flow of this system and, in particular the solution for the given initial conditions.

Problem 4: (15 pts) Solve the differential equation $\ddot{x} + 4\dot{x} + 5x = 0,\ \ x(0) = 6,\ \dot{x}(0) = 4$ two ways:

(a) using previous (non-matrix) methods.

(b) by reduction of order, turning this into a system of two 1st order ODEs, and solving using eigenvalue/eigenvector methods. Draw the phase plane with a range of sample trajectories, including the one associated with the given initial conditions.

Problem 5: (15 pts - <u>more challenging</u>): Solve the system $\left\{ \begin{aligned} \frac{dx}{dt} &= 3x \qquad\ \ - 4z \\ \frac{dy}{dt} &= 10x + 5y - 10z \\ \frac{dz}{dt} &= 6x + 2y - 7z \end{aligned} \right\}$ with initial conditions $x(0) = 1,\ y(0) = -1,\ z(0) = 1$. Express your solution in terms of <u>real-valued functions</u>. Sketch the general flow of this system and, in particular the solution for the given initial conditions.

Problem 6: (10 pts) Consider the system $\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$ with $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Sketch a direction field for **Ax** (or use the Java tool). Based on your sketch, describe the trajectories geometrically. Can you find the solutions analytically?

Problem 7: (10 pts) Let **A** be an $n \times n$ matrix and k a scalar. Consider the following two systems:

$$\left\{ \begin{aligned} \frac{d\mathbf{x}}{dt} &= \mathbf{A}\mathbf{x} && \text{(I)} \\ \frac{d\mathbf{c}}{dt} &= (\mathbf{A} + k\,\mathbf{I}_n)\mathbf{c} && \text{(II)} \end{aligned} \right\}$$

Show that if $\mathbf{x}(t)$ is a solution of system (I), then $\mathbf{c}(t) = e^{kt}\mathbf{x}(t)$ is a solution of system (II).

Problem 8: (10 pts) Find all solutions of the system $\frac{d\mathbf{x}}{dt} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}\mathbf{x}$, where λ is an arbitrary constant.

Hint: Problems 6 and 7 are helpful.