ORDINARY DIFFERENTIAL EQUATIONS

ODEs

Modeling Change

Continuous vs DISCRETE

Interest w/Compounding

Annual compounding   Rate R

Initial Amt $\underline{P_0}$

1 year $P_1 = P_0 + RP_0 = P_0(1+R)$

2 yrs $P_2 = P_1(1+R) = P_0(1+R)^2$

$\vdots$

t yrs $P_t = P(t) = P_0(1+R)^t$

Compound monthly

1 month $P = P_0\left(1+\frac{R}{12}\right)$

1 year $P(1) = P_0\left(1+\frac{R}{12}\right)^{12}$

t years $P(t) = P_0\left(1+\frac{R}{12}\right)^{12t}$

Compound n times per year

t yrs $P(t) = P_0\left(1+\frac{R}{n}\right)^{nt}$

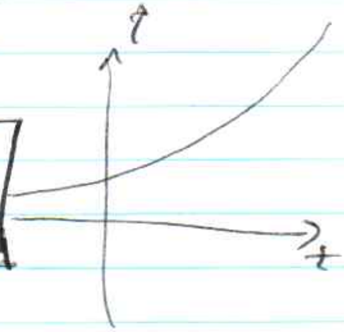
Continuous compounding

$P(t) = \lim_{n\to\infty} P_0\left(1+\frac{R}{n}\right)^{nt}$ $\longrightarrow$

$$P(t) = P_0 \left[\lim_{n\to\infty} \left(1+\frac{R}{n}\right)^{nt} \right]$$

$$= P_0 \Big[\underbrace{\lim_{n\to\infty} \left(1+\frac{R}{n}\right)^{n}}_{= \; e^{R}} \Big]^{t}$$

$$\Rightarrow \boxed{P(t) = P_0 e^{Rt}}$$



ALTERNATIVE — ODE's.

Natural Growth Model.

P = pop'n, \$, etc.

$$\boxed{\frac{dP}{dt} = RP}$$

Relative Growth Rate $\boxed{\dfrac{dP/dt}{P} = R}$

$$\frac{1}{P}\frac{dP}{dt} = R$$

ODE + Initial Condition (IC) $P(0) = P_0$

ODE + IC $\Rightarrow$ IVP Initial VALUE Problem

Existence and Uniqueness of solutions

General: $\dfrac{dP}{dt} = F(t, P)$

Ex) $\dfrac{dx}{dt} = at$ $\quad x(0) = x_0$ $\quad x(t) = \frac{1}{2}at^2 + C$

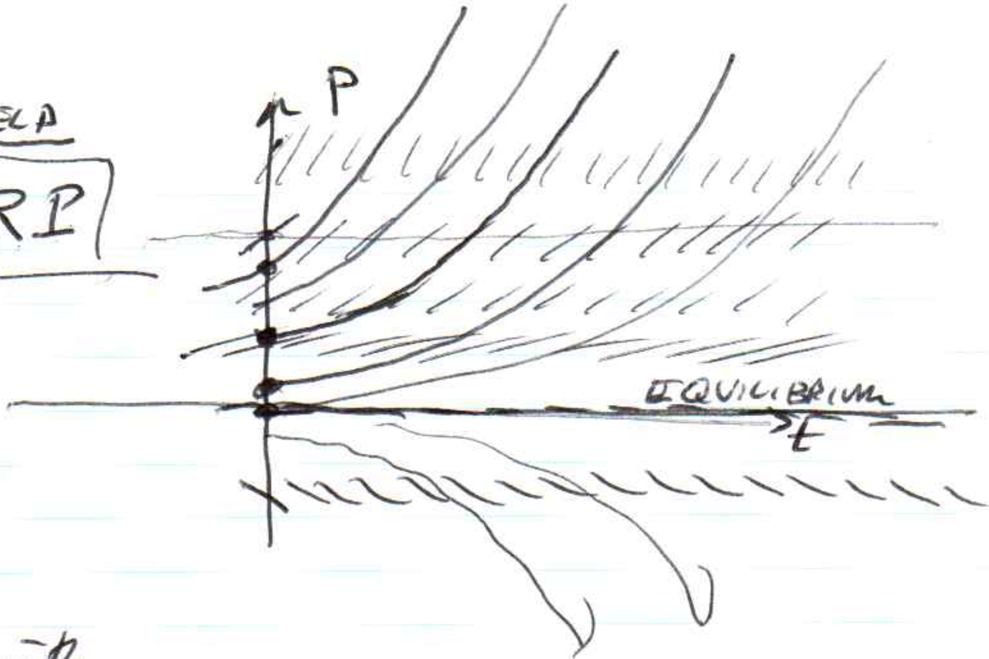
$x(0) = C = x_0$

$x(t) = x_0 + \frac{1}{2}at^2$

SLOPE FIELD

$$\boxed{\frac{dP}{dt} = RP}$$

$P(t) = ?$

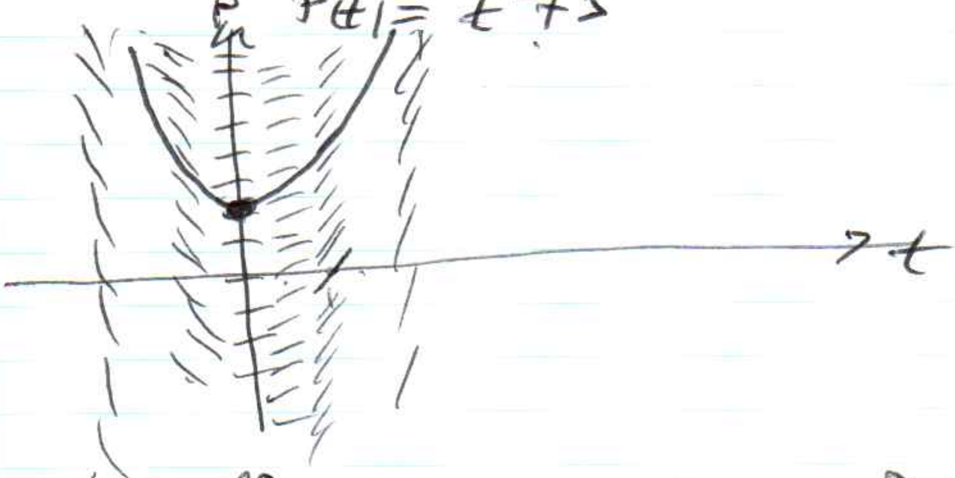


Compare with.

$$\frac{dP}{dt} = 2t, \quad P(0) = 5$$

$$P(t) = t^2 + C \qquad P(0) = C = 5$$

$$P(t) = t^2 + 5$$



Solve $\frac{dP}{dt} = RP$, $P(0) = P_0$ $\quad P(t)$

$$\frac{1}{P}\frac{dP}{dt} = R$$

$$\frac{d}{dt}\left[\ln(P)\right] = R$$

$$e^{\ln P} = e^{Rt + C}$$

$$\longrightarrow P(t) = Ae^{Rt}$$

$$P(0) = A = P_0$$

$$\boxed{P(t) = P_0 e^{Rt}}$$

$$\frac{dP}{dt} = RP, \quad P(0) = P_0$$

Separation of VARIABLES

$$\int \frac{dP}{P} = \int R\,dt$$

$$\ln|P| = Rt + C$$

$$e^{\ln|P|} = e^{Rt+C}$$

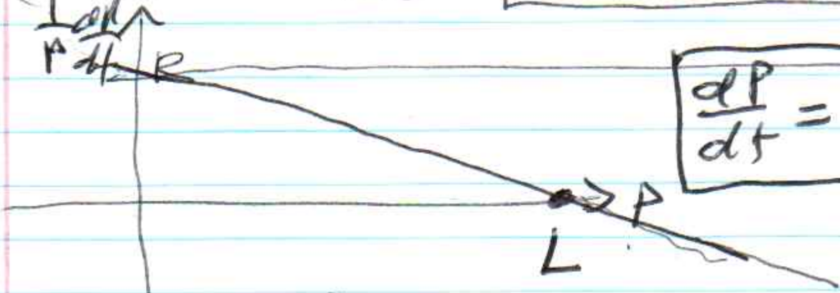
$$P(t) = Ae^{Rt} \quad \Rightarrow \quad \underline{P(t) = P_0 e^{Rt}}$$

$$P(0) = A = P_0$$

Other Growth Models

GROWTH in a limited environment
(LOGISTIC GROWTH model)

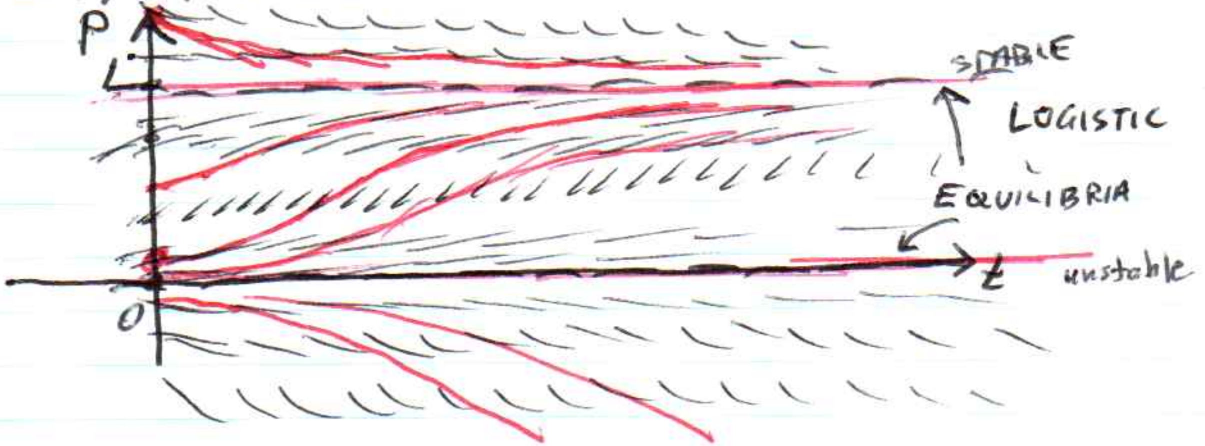
Relative growth Rate $$\boxed{\frac{1}{P}\frac{dP}{dt} = R\left(1 - \frac{P}{L}\right)}$$



$$\boxed{\frac{dP}{dt} = RP\left(1 - \frac{P}{L}\right)}$$

L = "carrying capacity"

SLOPE FIELD.



Analytic Solution

$$\frac{dP}{dt} = RP\left(1 - \frac{P}{L}\right) \qquad P(0) = P_0$$

Sep'n of Variable

$$\int \frac{dP}{P\left(1 - P/L\right)} = \int R\,dt$$

PARTIAL FRACTIONS

$$\Longrightarrow \boxed{P(t) = \frac{L P_0}{(L - P_0)e^{-Rt} + P_0}}$$

Derived via PARTIAL FRACTIONS and algebra.

Note: $\lim_{t \to \infty} P(t) = \frac{L P_0}{P_0} = L$

LINEAR nth order ODE's.

Form: $x(t)$, $\frac{dx}{dt}$, $\frac{d^2x}{dt^2}$, ... $\frac{d^nx}{dt^n}$

$$\frac{d^nx}{dt^n} + p_{n-1}(t)\frac{d^{n-1}x}{dt^{n-1}} + \cdots + p_1(t)\frac{dx}{dt} + p_0(t)x = g(t)$$

where $p_0(t)$, $p_1(t)$, ... are coefficient functions.

Note: Much easier if p_i's are all constant functions.

If $g(t) = 0$ (all t) $\to$ homogeneous.

If $g(t)$ not constant $\to$ inhomogeneous.

Ex: $$\frac{d^2x}{dt^2} + t\frac{dx}{dt} + 3x = t^3$$ 2nd order, inhomogeneous, Linear ODE

LINEAR? "Preserves scaling and adding"

$\frac{d}{dt} = D$ = "take the derivative"

$$\frac{d}{dt}\left[f(t) + g(t)\right] = f'(t) + g'(t)$$ preserves addition

$$\frac{d}{dt}\left[c f(t)\right] = c f'(t)$$ Preserves scaling

$$\frac{d}{dt}\left[c_1 x_1(t) + c_2 x_2(t)\right] = c_1 \dot{x}_1(t) + c_2 \dot{x}_2(t)$$

$\left(\dot{x}(t) = \frac{dx}{dt}\right)$ $\boxed{\text{D acts linearly}}$

$D^2 = \frac{d^2}{dt^2}$ "operators"

$= D \circ D$ also linear

$D^n = \frac{d^n}{dt^n}$ also linear.

$T = D^n + P_{n-1}(t) D^{n-1} + P_{n-2}(t) D^{n-2} + \cancel{\cdots} + P_1(t) D + P_0(t) I$

also linear. LINEAR COMBINATION OF LINEAR OPERATORS IS ALSO LINEAR.

$$\boxed{T(x(t)) = f(t)}$$

T is a linear operator

LINEARITY METHOD

Solve $\boxed{T(x(t)) = f(t)}$ $*$

Suppose $x_p(t)$ is just one particular solution to $*$, i.e. $T(x_p(t)) = f(t)$

Let $x(t)$ be any other solution to $*$.

Consider $x(t) - x_p(t)$

$T(x(t) - x_p(t)) = T(x(t)) - T(x_p(t))$

$= f(t) - f(t) = 0$

$\therefore$ $x(t) - x_p(t)$ is a homogeneous solution

$T(x(t) - x_p(t)) = 0$ all t

$x_h(t) = x(t) - x_p(t)$ $\Longrightarrow$

$$\boxed{x(t) = x_h(t) + x_p(t)}$$ gives general solution

<u>Method:</u>

① First, solve $T(x_h) = 0$

$$\Rightarrow \boxed{x_h(t)}$$ This will generally involve unknown constants.

② By hook or by crook, find just one particular soln to (*).

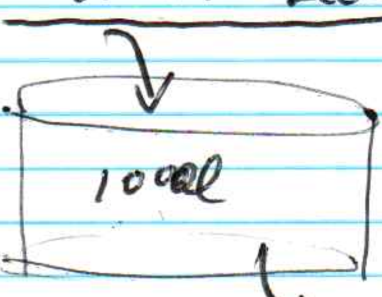
$$T(x_p(t)) = g(t) \qquad \boxed{x_p(t)}$$

③ <u>General solution</u> must be

$$x(t) = x_h(t) + x_p(t)$$

④ use IC's to find any constants

<u>MIXING PROBLEM</u> : 1000 l tank



Pour in saline at a rate of 10 l/hour with concentration of 50 gm/liter

STIR CONSTANTLY.

Let excess out

Initial concentration = 30 gm/liter

Let $x(t)$ = amt of salt at time t (in gms)

$$\frac{dx}{dt} = (\text{Rate IN}) - (\text{Rate Out})$$

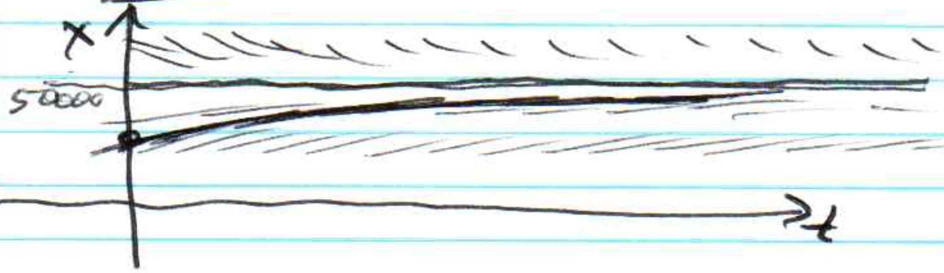
$$= \left(\frac{10\,\cancel{\ell}}{hr}\right)\left(\frac{50\,gm}{\cancel{\ell}}\right) - \left(\frac{10\,\cancel{\ell}}{hr}\right)\left(\frac{x(t)\,gm}{1000\,\cancel{\ell}}\right)$$

$$\boxed{\frac{dx}{dt} = 500 - .01X}$$

<u>ODE</u>

$$X(0) = (1000\,\ell)\,\frac{30\,gm}{\ell} = 30000\,gm$$

<u>X(t)</u>   How do we solve?



EQUILIBRIUM when $500 - .01x = 0$

$x = 50{,}000\,gm$

<u>Analytic Solution</u>

<u>Separation of VARIABLES</u> $\displaystyle\int \frac{-.01\,dx}{500 - .01x} = \int -.01\,dt$

$$e^{\ln|500 - .01x|} = e^{-.01t + C}$$

$$500 - .01x = A e^{-.01t}$$

$t = 0$, $x = \cancel{\ }$ $\quad 500 - 300 = A$

$x = 30000$ $\quad A = 200$

$$500 - .01x = 200e^{-.01t}$$

$$.01x = 500 - 200e^{-.01t}$$

$$\boxed{x(t) = 50000 - 20000e^{-.01t}}$$

LINEARITY METHOD

⓪ Put in correct form first

$$\frac{dx}{dt} = 500 - .01x$$

$$\boxed{\begin{array}{l} \frac{dx}{dt} + .01x = 500 \\ x(0) = 30000 \end{array}}$$

1st order linear const. coeff Inhomog ODE

① HOMOG $\frac{dx}{dt} + .01x = 0$

$$\frac{dx}{dt} = -.01x \rightarrow \boxed{x_h = Ce^{-.01t}}$$

② PART. $\frac{dx}{dt} + .01x = 500$ EQUILIBRIUM SOLUTION is a particular SOLUTION.

$$x_p(t) = 50000$$

③ $\therefore$ $x(t) = Ce^{-.01t} + 50000$

④ $x(0) = 30000$ $x(0) = C + 50000 = 30000$

$$C = -20000$$

$$\Rightarrow \boxed{x(t) = 50000 - 20000e^{-.01t}}$$

Solve 1st order Linear ODE

using an integrating factor.

see notes for details

$$\frac{dx}{dt} + p(t)x = q(t)$$

mult. both sides by $v(t)$ so that we can integrate.

$$\frac{dx}{dt} + px = q$$

$$\boxed{v\frac{dx}{dt} + vpx = qv}$$

Recall product Rule: $\frac{d}{dt}(vx) = v\frac{dx}{dt} + x\frac{dv}{dt}$

Set $x\frac{dv}{dt} = vpx$

$$\frac{dv}{dt} = vp \quad \text{separable}$$

$$\int \frac{dv}{v} = \int p\,dt$$

$$\ln v = \int p(t)\,dt$$

$$e \qquad e$$

$$v(t) = e^{\int p(t)\,dt}$$

THIS IS THE DESIRED INTEGRATING FACTOR

$$\frac{d}{dt}(\ vx) = g\,v$$

to be continued next week

Addendum: FOR ODE $\frac{dx}{dt} + .01x = 500$, (*)

$p(t) = .01$, $\int p(t)dt = \int .01dt = .01t$ (NO NEED FOR +C)

$\therefore$ integrating facta is $e^{\int p dt} = e^{.01t}$.

MULTIPLY BOTH SIDES OF (*) by integrating factor:

$$\underbrace{e^{.01t}\frac{dx}{dt} + .01e^{.01t}x}_{\frac{d}{dt}\left[e^{.01t}x\right]} = 500e^{.01t}$$

$$\frac{d}{dt}\left[e^{.01t}x\right] = 500e^{.01t}$$

Integrate both sides (don't forget the +C now!)

$$e^{.01t}\cdot x = \frac{500e^{.01t}}{.01} + C = 50000e^{.01t} + C$$

Solve for $x = x(t)$: (DIVIDE both sides by $e^{.01t}$)

$$x(t) = 50000 + Ce^{-.01t}$$

NOW use initial condition $x(0) = 30000$:

$$x(0) = 50000 + C = 30000 \Rightarrow C = -20000$$

$$\therefore \boxed{x(t) = 50000 - 20000e^{-.01t}}$$