LINEARITY in ODE'S

"$T(x(t))$"

$$\frac{d^n x}{dt^n} + p_{n-1}(t)\frac{d^{n-1}x}{dt^{n-1}} + \cdots + p_1(t)\frac{dx}{dt} + p_0(t)\,x = g(t)$$

LINEAR ODE

$\boxed{\text{I}}$ method: homog solns $x_h(t)$ (1)
part. soln $x_p(t)$ (2)

$$\Rightarrow \quad x(t) = x_h(t) + x_p(t) \quad (3)$$

Initial conditions $\longrightarrow$ $x(t)$ unique.

$\boxed{\text{II}}$ If $T(x(t)) = g_1(t) \to$ soln $x_1(t)$
$T(x(t)) = g_2(t) \to$ soln $x_2(t)$

Then solution to $T(x(t)) = c_1 g_1(t) + c_2 g_2(t)$

is $x(t) = c_1 x_1(t) + c_2 x_2(t)$

Ex: $\ddot{x} + 5\dot{x} + 4x = 3\underbrace{\cos 2t}_{g_1(t)} + 4\underbrace{t^2}_{g_2(t)}$

(a) solve $\ddot{x} + 5\dot{x} + 4x = \cos 2t \longrightarrow x_1(t)$

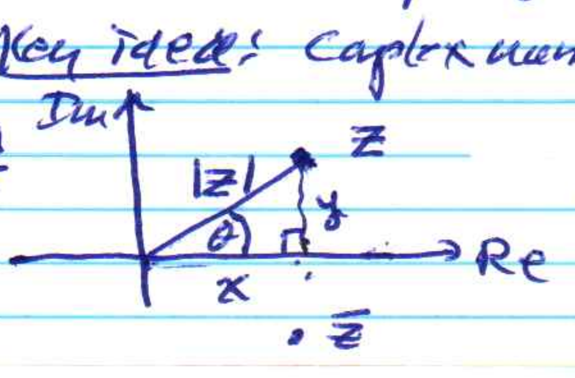
(b) solve $\ddot{x} + 5\dot{x} + 4x = t^2 \longrightarrow x_2(t)$

$$\Rightarrow \quad x(t) = 3\,x_1(t) + 4\,x_2(t)$$

Exponential Inputs, Sinusoidal Inputs

Key idea: complex numbers, complex functions.

$\mathbb{C}$



$z = x + iy \qquad i^2 = -1$

$\mathrm{Re}(z) = x \qquad \mathrm{Im}(z) = y$

modulus $z = |z| = \sqrt{x^2+y^2}$

$\tan\theta = y/x \qquad \theta = \tan^{-1}(y/x)$

ARG(z)

* Euler's Formula: $e^{it} = \cos t + i \sin t$

$$\left[\text{Euler's Identity}: \ e^{i\pi} = -1\right]$$

Algebra of cx number. ["cx" = complex]

$$Z_1 + Z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$$

$$Z_1 Z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$$

$$z = x + iy \qquad \bar{z} = x - iy$$

$$z\bar{z} = (x+iy)(x-iy) = x^2 + y^2 = |z|^2$$

* Recall Maclaurin Series.

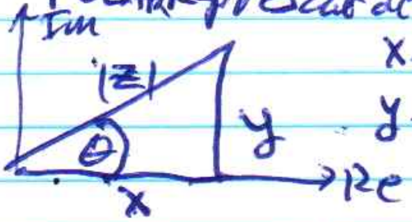
$$\left.\begin{aligned} e^t &= 1 + t + \frac{t^2}{2!} + \cdots + \frac{t^n}{n!} + \cdots \\ \cos t &= 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots \\ \sin t &= t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots \end{aligned}\right\} \text{absolutely convergent power series}$$

$$e^{it} = 1 + it + \frac{(it)^2}{2!} + \cdots + \frac{(it)^n}{n!} + \cdots$$

$$= \left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots\right) + i\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots\right)$$

$$\boxed{e^{it} = \cos t + i \sin t}$$

Polar representation



$x = |z| \cos\theta$

$y = |z| \sin\theta$

$$z = x + iy \quad \text{(RECT)}$$

$$= |z|(\cos\theta + i\sin\theta)$$

$$= |z| e^{i\theta}$$

POLAR FORM

Mult. in polar form [multiplication]

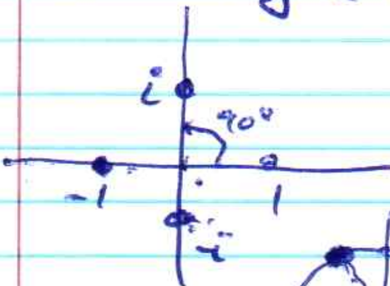
$$z_1 z_2 = \left(|z_1| e^{i\theta_1}\right)\left(|z_2| e^{i\theta_2}\right)$$

$$|z_1||z_2| e^{i(\theta_1+\theta_2)}$$

MULTIPLY THE MODULI
Add the arguments (angles)

$$|z_1 z_1| = |z_1||z_2|$$

$$\operatorname{Arg}(z_1 z_2) = \operatorname{Arg} z_1 + \operatorname{Arg} z_2$$



$$i^2 = i \cdot i = -1$$

$$i^3 = -i \qquad i^4 = +1$$

$$z^3 = 1$$

$$z^3 - 1 = 0$$

$$(z-1)(z^2+z+1) = 0$$

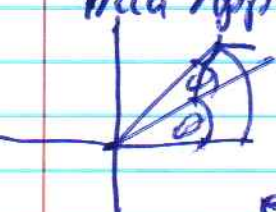
$$z = 1 \qquad z = \frac{-1 \pm \sqrt{-3}}{2}$$

$$= \frac{-1 \pm i\sqrt{3}}{2}$$

nth Roots of unity

$$z^n = 1$$

TRIG Application: Sum of angle formulas.



$$e^{i(\theta+\phi)} = \cos(\theta+\phi) + i\sin(\theta+\phi)$$

$$\|$$

$$e^{i\theta} e^{i\phi} = (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)$$

$$= (\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\cos\theta\sin\phi + \sin\theta\cos\phi)$$

$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$

$$\sin(\theta+\phi) = \cos\theta\sin\phi + \sin\theta\cos\phi$$

Calculus Application

$$\int e^{2t}\sin 3t\,dt$$

Integration Technique: Integ. by Parts 2x, solve the integrals.

$$\underline{\int e^{at}\cos bt\,dt} + i\underline{\int e^{at}\sin bt\,dt}$$

$$= \int e^{at}(\cos bt + i\sin bt)\,dt$$

$$= \int e^{at}e^{ibt}\,dt = \int e^{(a+ib)t}\,dt$$

$$\int e^{mt}\,dt = \frac{e^{mt}}{m} \qquad = \frac{e^{(a+ib)t}}{a+ib}\frac{(a-ib)}{(a-ib)}$$

$$= \frac{e^{at}(\cos bt + i\sin bt)(a-ib)}{a^2+b^2}$$

$$= \frac{e^{at}}{a^2+b^2}\left[(a\cos bt + b\sin bt) + i(a\sin bt - b\cos bt)\right]$$

$$\int e^{at}\cos bt\,dt = \frac{e^{at}}{a^2+b^2}(a\cos bt + b\sin bt) + C$$

$$\int e^{at}\sin bt\,dt = \frac{e^{at}}{a^2+b^2}(a\sin bt - b\cos bt) + C$$

ODE Applications $\quad \omega^2 = k/m$

$\boxed{\text{I}}$ Solve $\ddot{x} + \omega^2 x = 0$

Try exp soln $x = e^{rt}$

$\dot{x} = re^{rt} \quad \ddot{x} = r^2 e^{rt}$

$r^2 e^{rt} + \omega^2 e^{rt} = 0$

$(r^2 + \omega^2) e^{rt} = 0 \qquad p(r) = r^2 + \omega^2$

$p(r) = r^2 + \omega^2 = 0 \implies r = \pm i\omega$

Formal solns $\{e^{i\omega t}, e^{-i\omega t}\}$

$x(t) = c_1 e^{i\omega t} + c_2 e^{-i\omega t}$

$= c_1(\cos\omega t + i\sin\omega t) + c_2(\cos\omega t - i\sin\omega t)$

$= \underbrace{(c_1 + c_2)}_{a}\cos\omega t + \underbrace{i(c_1 - c_2)}_{b}\sin\omega t$

$a\cos\omega t + b\sin\omega t$

$\text{Span}\{\cos\omega t, \sin\omega t\}$

$\boxed{\text{II}}$ Complex Replacement

Solve $\ddot{x} + 5\dot{x} + 4x = 4\cos 3t$

Could try: $x_p = a\cos 3t + b\sin 3t$

MOTIVATIONS

Instead: Solve $\ddot{x} + 5\dot{x} + 4x = 4e^{3t}$

Try $x = ce^{3t} \qquad 28ce^{3t} = 4e^{3t}$

$\dot{x} = 3ce^{3t} \qquad 28c = 4$

$\ddot{x} = 9ce^{3t} \qquad c = 1/7$

Exponential Response Formula (ERF)

$$[p(D)]\, x(t) = a e^{rt}$$

Try $x = c e^{rt}$ and subst into ODE

$$\Rightarrow \underline{c\, p(r)} e^{rt} = \underline{a} e^{rt}$$

$$c \cdot p(r) = a \Rightarrow c = \frac{a}{p(r)}$$

$$\Rightarrow \boxed{x_p(t) = \frac{a e^{rt}}{p(r)}}$$ EXP RESPONSE FORMULA.

ex: $\ddot{x} + 5\dot{x} + 4x = 4e^{3t}$

$p(s) = s^2 + 5s + 4 \quad a = 4 \quad r = 3$

$$\therefore\ x_p(t) = \frac{a e^{rt}}{p(r)} = \frac{4e^{3t}}{28} = \frac{e^{3t}}{7}$$

$p(r) = 3^2 + 5 \cdot 3 + 4 = 28$

Note: Only good if r is not a characteristic Root.

Note: In that case, we have Resonant Response Formulas.

If $p(r) = 0$

$$x_p(t) = \frac{a t e^{rt}}{p'(r)}, \quad \frac{a t^2 e^{rt}}{p''(r)}, \text{ etc.}$$

(more on these at a later date)

Sinusoidal inputs.

Solve $\ddot{x} + 5\dot{x} + 4x = 3\cos 4t$ ①

$\ddot{x} + 5\dot{x} + 4x = 3\sin 4t$ ②

$\ddot{y} + 5\dot{y} + 4y = 3\sin 4t$

Let $z = x + iy$ complex replacement

$\dot{z} = \dot{x} + i\dot{y}$

$\ddot{z} = \ddot{x} + i\ddot{y}$ $\qquad p(s) = s^2 + 5s + 4$

$\ddot{z} + 5\dot{z} + 4z$

$= (\ddot{x} + 5\dot{x} + 4x) + i(\ddot{y} + 5\dot{y} + 4y)$

$3(\cos 4t + i\sin 4t) = 3e^{i4t}$

Solve $\ddot{z} + 5\dot{z} + 4z = 3e^{4it}$

easy to solve use ERF

$$z = \frac{3e^{4it}}{p(4i)} = \boxed{\frac{3e^{4it}}{4(-3+5i)}}$$

$p(4i) = -16 + 20i + 4$

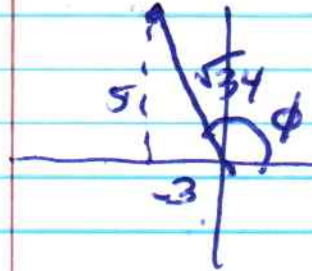
$= -12 + 20i = 4(-3+5i)$

$\boxed{A} = \frac{3}{4}\frac{(\cos 4t + i\sin 4t)(-3-5i)}{(-3+5i)(-3-5i)}$

$= \frac{3}{4\cdot 34}\left[(-3\cos 4t + 5\sin 4t) + i(-3\sin 4t - 5\cos 4t)\right]$

① $\frac{3}{136}(-3\cos 4t + 5\sin 4t)$ ② $\frac{3}{136}(-3\sin 4t - 5\cos 4t)$

$\boxed{B}$ $Z = \dfrac{3e^{4it}}{4(-3+5i)} = \boxed{\dfrac{3e^{4it}}{4\sqrt{34}\,e^{i\phi}}}$



$\tan\phi = -\frac{5}{3}$

$-3+5i = \sqrt{34}\,e^{i\phi}$

$= \dfrac{3}{4\sqrt{34}}\,\underbrace{e^{i(4t-\phi)}}_{\cos(4t-\phi)+i\sin(4t-\phi)}$

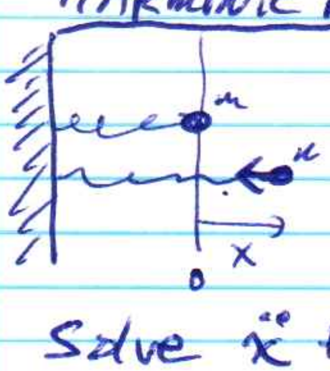
$= \dfrac{3}{4\sqrt{34}}\left[\cos(4t-\phi) + i\sin(4t-\phi)\right]$

amplitude

phase angle

Amplitude - phase

HARMONIC RESPONSE



$\ddot{x} + \frac{k}{m}x = 0$ Hooke's Laws

no force, no input

$\frac{k}{m} = \omega^2$

$\omega = \sqrt{\frac{k}{m}}$

Solve $\ddot{x} + \omega^2 x = a\cos\omega_0 t$ $\quad \omega_0 \neq \omega$

① homog soln $\ddot{x} + \omega^2 x = 0$

$\Rightarrow x_h(t) = c_1\cos\omega t + c_2\sin\omega t$

② PART soln: CX Replacement

$\ddot{z} + \omega^2 z = a e^{i\omega_0 t}$ Recover Real part

ERF $\quad p(s) = s^2 + \omega^2 \qquad p(i\omega_0) = \omega^2 - \omega_0^2$

$$z_p = \frac{a e^{i\omega_0 t}}{p(i\omega_0)} = \frac{a e^{i\omega_0 t}}{\omega^2 - \omega_0^2}$$

Real Part: $\boxed{\frac{a}{\omega^2-\omega_0^2} \cos \omega_0 t = x_p(t)}$

Solve $\frac{dx}{dt} + 2x = e^{-t} \cos 3t$.

Homog $\quad \frac{dx}{dt} + 2x = 0 \quad \frac{dx}{dt} = -2x$

$$x_h(t) = C e^{-2t}$$

Particular: use complex Replacement

$$z = x + iy$$

$$\frac{dz}{dt} + 2z = e^{-t} e^{3it} = e^{(-1+3i)t}$$

$$p(s) = s + 2 \qquad p(-1+3i) = 1 + 3i$$

ERF: $\quad z_p(t) = \frac{e^{(-1+3i)t}}{p(-1+3i)} = \frac{e^{(-1+3i)t}}{1+3i}$

$$z_p(t) = \frac{e^{-t} e^{3it}}{1+3i} = \frac{e^{-t}(\cos 3t + i \sin 3t)(1-3i)}{(1+3i)(1-3i)}$$

$$= \frac{e^{-t}}{10}\left[(\cos 3t + 3 \sin 3t) + i\,(\text{who cares})\right]$$

$$\therefore x_p(t) = \frac{e^{-t}}{10}(\cos 3t + 3 \sin 3t)$$

$$= \frac{e^{-t}}{10} \sqrt{10} \cos(3t - \phi)$$