Autonomous (time-independent) 1st order ODES

$$\frac{dx}{dt} = F(t,x) \quad \text{general}$$

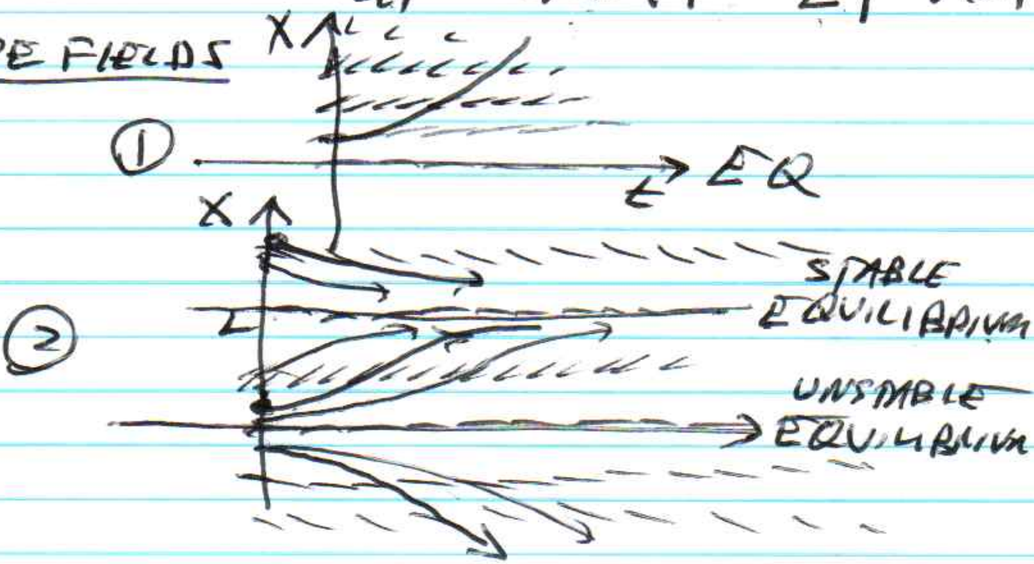
$$\frac{dx}{dt} = F(x) \quad \text{Autonomous.}$$

Ex: Natural growth ① $\frac{dx}{dt} = kx$, $x(0) = x_0$

$\Rightarrow x(t) = x_0 e^{kt}$

Logistic Model ② $\frac{dx}{dt} = kx\left(1 - \frac{x}{L}\right) = F(x)$

SLOPE FIELDS



EQUILIBRIUM $\frac{dx}{dt} = 0$

DERIVATIVE TEST FOR STABILITY

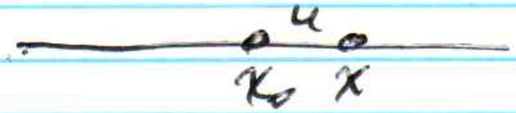
$\frac{dx}{dt} = F(x)$, autonomous ; $F(x)$ differentiable

Linear Approx x_0 EQUILIBRIUM

$$F(x) \cong F(x_0) + F'(x_0)(x - x_0)$$

x_0 EQUILIBRIUM $\left.\frac{dx}{dt}\right|_{x_0} = F(x_0) = 0$

$$\frac{dx}{dt} = F(x) \cong \cancel{F(x_0)} + F'(x_0)(x - x_0)$$



$u = x - x_0$

$\frac{du}{dt} = \frac{dx}{dt}$

$$\boxed{\frac{du}{dt} \cong F'(x_0) \cdot u}$$

$F'(x_0) > 0$ growth

$F'(x_0) < 0$ decay

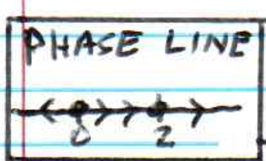
$F'(x_0) = 0$?

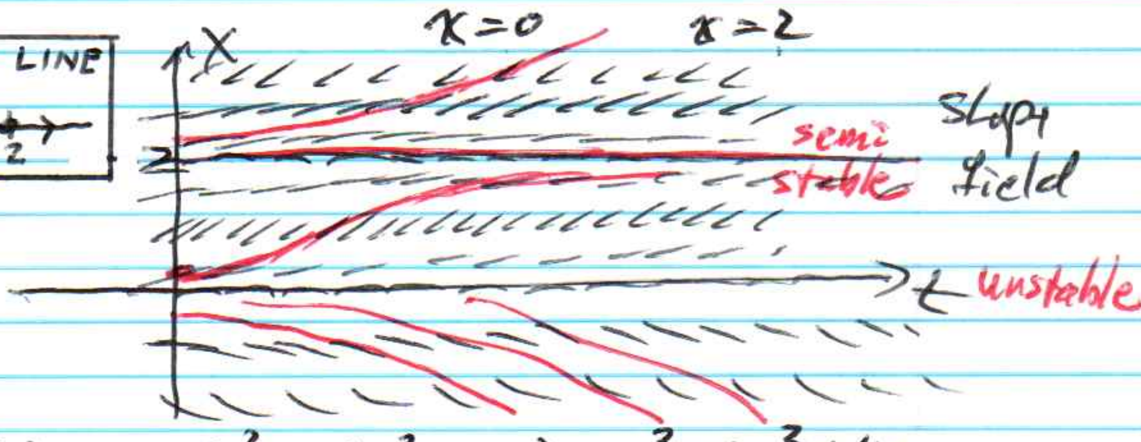
Example: $\frac{dx}{dt} = x(x-2)^2 = F(x)$ Autonomous

Can solve this! $\int \frac{dx}{x(x-2)^2} = \int dt$ $\underline{\underline{x(t)}}$

PARTIAL FRACTIONS

EQUILIBRIA? $F(x) = x(x-2)^2 = 0$





$F(x) = x(x-2)^2 = x(x^2-4x+4) = x^3 - 4x^2 + 4x$

$F'(x) = 3x^2 - 8x + 4$ $F'(0) = 4 > 0$ unstable

$F'(2) = 12 - 16 + 4 = 0$?

nth order Linear ODE's.

$$\underbrace{\frac{d^n x}{dt^n} + P_{n-1}(t)\frac{d^{n-1}x}{dt^{n-1}} + \cdots + P_1(t)\frac{dx}{dt} + P_0(t)\cdot x}_{\text{system}} = \underset{\text{Input}}{q(t)}$$

Response $x(t)$ / Solution

1st order: • Sep'n of variables
• Integrating factor

nth order • $\boxed{\text{LINEARITY}}$

(a) Homog (b) PART. SOLN → (c) General

IC's → unique soln.

Example: $\frac{dx}{dt} + t x = 2t$ general soln.

Integ factor $p(t) = t$ $\boxed{e^{\int p\,dt}}$

$$\int t\,dt = \frac{t^2}{2}$$

$$e^{\frac{1}{2}t^2}$$

$$\underbrace{e^{\frac{1}{2}t^2}\frac{dx}{dt} + t\cdot e^{\frac{1}{2}t^2}x}_{\frac{d}{dt}\left[e^{\frac{1}{2}t^2}x\right]} = 2te^{\frac{1}{2}t^2}$$

$$\frac{d}{dt}\left[e^{\frac{1}{2}t^2}x\right] = 2te^{\frac{1}{2}t^2}$$

$$e^{\frac{1}{2}t^2}x = 2e^{\frac{1}{2}t^2} + C$$

$$\boxed{x(t) = \underset{\substack{\text{part} \\ \text{(steady-state)}}}{2} + \underset{\text{homog (transient)}}{Ce^{-\frac{1}{2}t^2}}}$$

Finding PARTICULAR SOLUTIONS.

1. method of Undetermined Coefficients
2. VARIATION OF PARAMETERS (VOP)

1st order VERSION

SITUATION : $\boxed{\frac{dx}{dt} + p(t)\, x = g(t)}$ (*)

1. Find homogeneous solution $x_h(t)$

$$\frac{dx}{dt} + p(t)x = 0 \quad \text{separable}$$

$$\frac{dx}{dt} = -p(t)x$$

$$\int \frac{dx}{x} = -\int p(t)dt$$

$$e^{\ln|x|} = e^{-\int p(t)dt + C}$$

$$x_h(t) = A e^{-\int p\,dt}$$

$x_h(t) = A\, \underline{x_1(t)}$ $\quad$ $x_1(t)$ is a basis for all homog. solns.

2. Find particular solution to (*)

Seek $x_p(t) = v(t)\,x_1(t) = v x_1$ $\quad$ "VARY THE PARAMETER"

Plug it into (*):

$$\frac{dx}{dt} = v\frac{dx_1}{dt} + x_1\frac{dv}{dt}$$

$$v\frac{dx_1}{dt} + x_1\frac{dv}{dt} + p v x_1 = g$$

$$v\underbrace{\left(\frac{dx_1}{dt} + p x_1\right)}_{=0} + \boxed{x_1\frac{dv}{dt} = g(t)}$$

$$x_1(t)\frac{dv}{dt} = g(t) \longrightarrow \boxed{\frac{dv}{dt} = \frac{g(t)}{x_1(t)}}$$

$$v(t) = \int \frac{g(t)}{x_1(t)}\,dt \implies \underline{x_p(t) = v(t)\,x_1(t)}$$

Ex: $\frac{dx}{dt} + tx = 2t$

Homog $\frac{dx}{dt} + tx = 0 \longrightarrow x_h(t) = A\underbrace{e^{-\frac{1}{2}t^2}}_{x_1(t)}$

$$v(t) = \int \frac{2t}{e^{-\frac{1}{2}t^2}}\,dt = \int 2te^{\frac{1}{2}t^2}\,dt$$

$$= 2e^{\frac{1}{2}t^2}$$

$$\therefore\ x_p(t) = vx_1 = 2\cancel{e^{\frac{1}{2}t^2}}\cdot\cancel{e^{-\frac{1}{2}t^2}} = \underline{\underline{2}}$$

Ex: $\frac{dx}{dt} + \frac{5}{t}x = 7t$ general soln.

① Homog $\frac{dx}{dt} + \frac{5}{t}x = 0$ $\quad \frac{dx}{dt} = -\frac{5x}{t}$

$$\int\frac{dx}{x} = -\int 5\frac{dt}{t} \qquad \ln|x| = -5\ln t + C$$

$$x_h(t) = A\,t^{-5} = \boxed{\frac{A}{t^5} = x_h(t)}$$

② PART (VOP): $x_1(t) = t^{-5}$ $\quad$ Seek $x_p = vx_1$

$$\frac{dv}{dt} = \frac{7t}{t^{-5}} = 7t^6 \rightarrow v(t) = t^7$$

$$x_p = t^7\cdot t^{-5} = t^2$$

③ general soln

$$x(t) = x_h(t) + x_p(t)$$

$$x(t) = \frac{A}{t^5} + t^2$$

If $x(1) = 4$ (IC), $x(1) = A + 1 = 4$

$A = 3$

$$\boxed{x(t) = \frac{3}{t^5} + t^2}$$

SINUSOIDAL INPUTS

Solve $\ddot{x} + 3\dot{x} + 2x = 5\sin t$ $\left[\begin{array}{l} x(0) = 1 \\ \dot{x}(0) = 2 \end{array}\right]$

IVP

LINEAR

① Homog $\ddot{x} + 3\dot{x} + 2x = 0$

Try exponential solutions $x(t) = e^{rt}$

$$\dot{x}(t) = re^{rt} \quad \ddot{x}(t) = r^2 e^{rt}$$

$$\underbrace{(r^2 + 3r + 2)}_{p(r)} e^{rt} = 0 \text{ for all } t$$

$p(r)$ characteristic polynomial.

$$p(r) = r^2 + 3r + 2 = (r+2)(r+1) = 0$$

$r = -2$ or $r = -1$ $\{e^{-2t}, e^{-t}\}$

Basis for all homog solns.

$$x_h(t) = c_1 e^{-2t} + c_2 e^{-t}$$

Q: Do these give all homog. solutions?

$\ddot{x} + 3\dot{x} + 2x = 0$ $\qquad D = \frac{d}{dt}$ LINEAR OPERATOR

$$(\underbrace{D^2 + 3D + 2I}_{p(D)})\, x(t) = 0 \quad \ddagger \qquad [p(D)]\, x(t) = 0$$

$p(r) = (r+2)(r+1)$ "FACTOR THE OPERATOR"

$p(D) = (D + 2I) \circ (D + I)$

True? $(D + 2I) \circ (D + I)\, x(t)$

$= (D + 2I)(\dot{x} + x)$

$= \ddot{x} + \dot{x} + 2\dot{x} + 2x$

$= \ddot{x} + 3\dot{x} + 2x$

$$\ddagger \quad (D + 2I)\Big[\underbrace{(D + I)\, x(t)}_{u(t)}\Big] = 0$$

$\dot{u} + 2u = 0 \qquad \frac{du}{dt} = -2u$

$\Rightarrow u(t) = c_1 e^{-2t}$

$\frac{dx}{dt} + x = c_1 e^{-2t}$ 1st order, linear inhomog

Integrating Factor $p(t) = 1 \qquad e^{\int p\,dt} = e^{t}$

$$\underbrace{e^{t}\frac{dx}{dt} + xe^{t}}_{\frac{d}{dt}(e^{t}x)} = c_1 e^{-t}$$

$\frac{d}{dt}(e^{t}x) = c_1 e^{-t} \qquad e^{t}x = -c_1 e^{-t} + c_2$

$x(t) = -c_1 e^{-2t} + c_2 e^{-t}$

Input Amplitude

PARTICULAR SOLN: $\ddot{x} + 3\dot{x} + 2x = 5\sin t$

$x(0) = 1, \ \dot{x}(0) = 2$

$x_h(t) = c_1 e^{-2t} + c_2 e^{-t}$

Try (2) $x_p(t) = a\cos t + b\sin t$

(3) $\dot{x}_p = b\cos t - a\sin t$

(1) $\ddot{x} = -a\cos t - b\sin t$

$(2a + 3b - a)\cos t + (2b - 3a - b)\sin t = 5\sin t$

$(a + 3b)\cos t + (-3a + b)\sin t = 5\sin t$

$$\left\{ \begin{array}{l} a + 3b = 0 \\ -3a + b = 5 \end{array} \right\} \quad \begin{array}{l} a = -3b \\ 9b + b = 10b = 5 \end{array} \quad \begin{array}{l} a = -\frac{3}{2} \\ b = \frac{1}{2} \end{array}$$

$x_p(t) = -\frac{3}{2}\cos t + \frac{1}{2}\sin t$

$x(t) = c_1 e^{-2t} + c_2 e^{-t} - \frac{3}{2}\cos t + \frac{1}{2}\sin t$

$\dot{x}(t) = -2c_1 e^{-2t} - c_2 e^{-t} + \frac{3}{2}\sin t + \frac{1}{2}\cos t$

$x(0) = c_1 + c_2 - \frac{3}{2} = 1$ $\qquad c_1 + c_2 = 5/2$

$\dot{x}(0) = -2c_1 - c_2 + \frac{1}{2} = 2$ $\qquad -2c_1 - c_2 = 3/2$

$-c_1 = 4 \quad c_1 = -4$

$c_2 = \frac{5}{2} + 4 = \frac{13}{2} = c_2$

$$x(t) = \underbrace{-4e^{-2t} + \frac{13}{2}e^{-t}}_{\text{transient}} \underbrace{- \frac{3}{2}\cos t + \frac{1}{2}\sin t}_{\text{steady-state}}$$

TRIGONOMETRY + Amplitude-phase form

$$a\cos\omega t + b\sin\omega t = A\cos(\omega t - \phi_0)$$

amplitude (A) phase angle (ϕ_0)

Recall

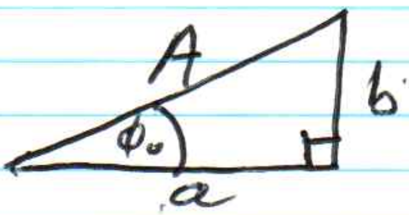
$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$

$$A\cos(\omega t-\phi_0) = A\left[\cos\omega t\cos(-\phi_0) - \sin\omega t\sin(-\phi_0)\right]$$

$$= \underbrace{(A\cos\phi_0)}_{=}\cos\omega t + \underbrace{(A\sin\phi_0)}_{=}\sin\omega t$$

$$= a\cos\omega t + b\sin\omega t$$

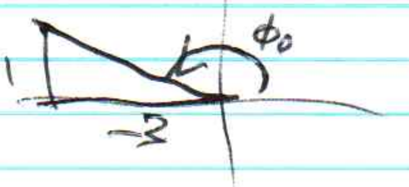
$$\left\{\begin{array}{l} A\cos\phi_0 = a \\ A\sin\phi_0 = b \end{array}\right\}$$



$$A = \sqrt{a^2+b^2}$$

$$\tan\phi_0 = \frac{b}{a}$$

$$x_p(t) = -\frac{3}{2}\cos t + \frac{1}{2}\sin t = \frac{1}{2}(-3\cos t + \sin t)$$



$$\tan\phi_0 = -\frac{1}{3} \rightarrow \phi_0$$

$$\sqrt{9+1} = \sqrt{10}$$

$$\Rightarrow x_p(t) = \frac{\sqrt{10}}{2}\cos(t-\phi_0)$$

Response Amplitude

$$\text{GAIN} = \frac{\text{Response Amplitude}}{\text{Input Amplitude}} = \frac{\sqrt{10}/2}{5} = \frac{\sqrt{10}}{10} = \frac{1}{\sqrt{10}}$$