EXISTENCE, UNIQUENESS of SOLNS to ODE'S.

$$\frac{dy}{dx} = F(x,y)$$ 1st order ODE

SLOPE FIELD

$y(x)$

$y(x_0) = y_0$

(x_0, y_0)

transverse crossing

impossible

non-unique solutions (tangent)

SEE LECTURE NOTES #1, #2

FIRST ORDER, LINEAR ODE'S.

$$\boxed{\frac{dx}{dt} + p(t)\,x = q(t)}$$ inhomogeneous

$x(0)$

METHODS OF SOLUTION

1. SEPARATION OF VARIABLES
2. LINEARITY → homog soln → $x_h(t)$; → part soln → $x_p(t)$

general solution $x(t) = x_h(t) + x_p(t)$

INITIAL CONDITIONS ⟹ unique soln $x(t)$

$x(0) = x_0$

$x(t_0) = x_0$

3. INTEGRATING FACTOR ⟹

Integrating factor for 1st order linear ODEs.

$$\frac{dx}{dt} + p(t)\,x = q(t)$$

Can always be solved.

$$\boxed{v(t)} \Rightarrow \boxed{v\frac{dx}{dt} + p v x} = v q = v(t)\,q(t)$$

TRICK! PRODUCT RULE

$$\frac{d}{dt}(vx) = \boxed{v\frac{dx}{dt} + \frac{dv}{dt}\cdot x}$$

Demand that $\frac{dv}{dt}x = p v x \rightarrow \boxed{\frac{dv}{dt} = pv}$

Solve using sep'n of VARIABLES

$$\int\frac{dv}{v} = \int p\,dt \Rightarrow \ln|v| = \int p(t)\,dt$$

$$e \qquad e$$

$$\boxed{v(t) = e^{\int p\,dt}}$$ Integrating Factor

$$\frac{d}{dt}\left[v(t)\,x(t)\right] = q(t)\,e^{\int p\,dt} \quad \text{etc.}$$

$\boxed{\begin{matrix} t = t_0 \\ x = x_0 \end{matrix}}$ $\cancel{v(t)\,x(t) = \int q(t)}$

(t_0, x_0)

$$v(t)\,x(t) - v(t_0)\,x(t_0) = \int_{t_0}^{t} q(s)\,e^{\int p\,ds}\,ds$$

$$\underline{\underline{v(t)\,x(t)}} = v(t_0)\,x(t_0) + \int_{t_0}^{t} q(s)\,e^{\int p\,ds}\,ds$$

$$x(t) = e^{-\int p\,dt}\left[v(t_0)\, x(t_0) + \int_{t_0}^{t} q(s)\, e^{\int p\, ds}\, ds \right]$$

Example ①: $\dfrac{dx}{dt} + 3x = 2t \qquad x(0) = 4$

LINEARITY

Homog $\quad \dfrac{dx}{dt} + 3x = 0 \quad \dfrac{dx}{dt} = -3x$

$$\Rightarrow \boxed{x_h(t) = c\, e^{-3t}}$$

PART $\quad x(t) = at + b$

$$\frac{dx}{dt} = a$$

$$\frac{dx}{dt} + 3x = a + 3(at+b) = 3at + (a+3b) = 2t$$

$$3a = 2 \qquad a + 3b = 0$$

$$a = 2/3 \qquad b = -\frac{a}{3} = -\frac{2}{9}$$

$$\boxed{x_p(t) = \frac{2}{3}t - \frac{2}{9}}$$

General $\quad x(t) = c\, e^{-3t} + \dfrac{2}{3}t - \dfrac{2}{9}$

$$x(0) = c - \frac{2}{9} = 4 \rightarrow c = 4\tfrac{2}{9} = \frac{38}{9}$$

$$\Rightarrow \boxed{x(t) = \frac{38}{9} e^{-3t} + \frac{2}{3}t - \frac{2}{9}}$$

Using Integrating factor

$$\boxed{\frac{dx}{dt} + 3x = 2t} \qquad x(0) = 4$$

$$p(t) = 3 \qquad \int p\,dt = \int 3\,dt = 3t$$

$$v = e^{\int p\,dt} = e^{3t}$$

$$\underbrace{e^{3t}\frac{dx}{dt} + 3e^{3t}x}_{\frac{d}{dt}\left[e^{3t}x\right]} = 2te^{3t}$$

$$\frac{d}{dt}\left[e^{3t}x\right] = 2te^{3t}$$

$$* \; e^{3t}x(t) = \int \underline{\underline{2t}}\,\underline{\underline{e^{3t}}}\,dt$$

INTEGRATION BY PARTS

LIPET

Logs
Invers
Polyi
Expt
Trig

$$\int u\,dv = uv - \int v\,du$$

$$u = 2t \qquad dv = e^{3t}dt$$

$$du = 2dt \qquad v = \tfrac{1}{3}e^{3t}$$

$$\int 2te^{3t}dt = \tfrac{2}{3}te^{3t} - \tfrac{2}{3}\int e^{3t}dt$$

$$= \tfrac{2}{3}te^{3t} - \tfrac{2}{9}e^{3t}$$

$$= \left(\tfrac{2}{3}t - \tfrac{2}{9}\right)e^{3t} + C$$

$$* \; e^{3t}x(t) = \left(\tfrac{2}{3}t - \tfrac{2}{9}\right)e^{3t} + C$$

$$x(t) = \tfrac{2}{3}t - \tfrac{2}{9} + Ce^{-3t}$$

etc.

EXAMPLE 2: $\frac{dx}{dt} + 3t^2 x = e^{-t}$ $x(0) = 1$

LINEARITY not so good for this example. [Try it!]

Integrating Factor $p(t) = 3t^2$

$$\int p\,dt = \int 3t^2\,dt = t^3 \qquad v(t) = e^{t^3}$$

$$\underbrace{e^{t^3}\frac{dx}{dt} + 3t^2 e^{t^3} x}_{} = e^{-t}e^{t^3} = e^{t^3-t}$$

$$\frac{d}{dt}\left[e^{t^3} x\right] = e^{t^3-t}$$

Integ both sides fr $t=0$ $t=t$

$$e^{t^3} x(t) - 1 = \int_{s=0}^{s=t} \left(e^{s^3-s}\right) ds$$

$$e^{t^3} x(t) = 1 + \int_0^t \left(e^{s^3-s}\right) ds$$

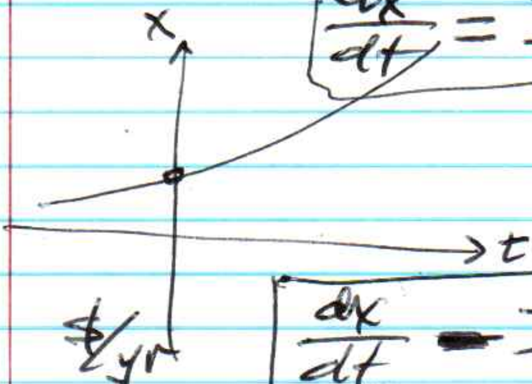
$$x(t) = e^{-t^3}\left[1 + \int_0^t e^{s^3-s}\,ds\right]$$

Input-Response Formulation for Linear ODE'S.
(Signal) $\rightarrow$ output.

<u>BANKING</u>   NATURAL exponential growth

$$\boxed{\frac{dx}{dt} = Ix}$$

I = interest rate



modify to acct for deposits/withdrawals.

$$\boxed{\frac{dx}{dt} - Ix = 0}$$

homogeneous.

$$\frac{dx}{dt} = Ix(t) + f(t)$$

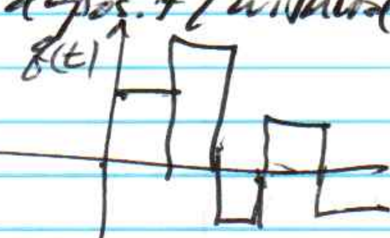
$$\boxed{\underbrace{\frac{dx}{dt} - Ix}_{\text{BANK}} = f(t)}$$

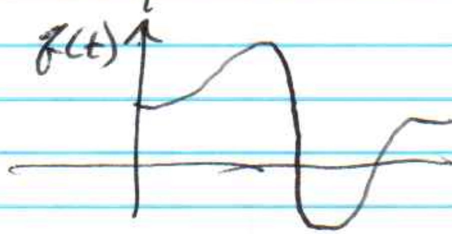
BANK

"system"

$f(t)$: Input (Signal)

Rate of deposit/withdrawal.





DIFFUSION, Newton's Law of Cooling.

$\boxed{y(t)}$ given

exterior temperature.

x(t) interior temperature

$$\frac{dx}{dt} = F(y-x)$$

$$\boxed{\frac{dx}{dt} = k(y-x)}$$

$k > 0$ coupling constant.

$$\frac{dx}{dt} = k\,y(t) - kx$$

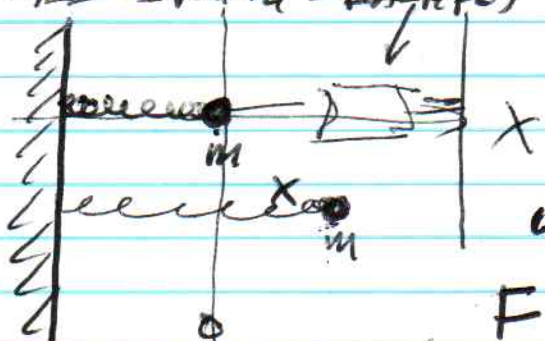
$$\boxed{\underbrace{\frac{dx}{dt} + kx}_{\text{System}} = k\,\underset{\uparrow}{y(t)}}$$

Linear, 1st order Inhomog ODE

System Input $x(0) = x_0$

$\rightarrow x(t)$ Response

MASS-SPRING-DASHPOT system, Hooke's Law.



$$\boxed{F = -kx}$$

w/ FRICTION v

$$F = -kx - cv + F_{ext}$$

w/ TOMFOOLERY $v = \frac{dx}{dt}$

Newton 2nd: $F = ma$ $a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$

$$ma = -kx - cv + F_{ext}$$

$$m\frac{d^2x}{dt^2} = -kx - c\frac{dx}{dt} + F_{ext}$$

$\dot{x} = \frac{dx}{dt}$ $\ddot{x} = \frac{d^2x}{dt^2}$

$$m\ddot{x} = -kx - c\dot{x} + F_{ext}$$

$$m\ddot{x} + c\dot{x} + kx = F_{ext}$$

$$\boxed{\underbrace{\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x}_{\text{System}} = \frac{F_{ext}}{m}}$$

2nd order linear ODE, Inhomogeneous

Input (me)

$x(t)$ Response