Rule (ESR)

EXPONENTIAL SHIFT FORMULA (ESF)

Solve $\ddot{x} + 3\dot{x} + 2x = \underline{t^2 e^{3t}}$ [MOST OF CLASS from Lecture #6 Notes]

Meth of undet coeffs

Homog: $p(s) = s^2 + 3s + 2 = (s+2)(s+1) = 0$

$s = -2,\ s = -1$

$x_h(t) = c_1 e^{-2t} + c_2 e^{-t}$

PART: $x_p(t) = (at^2 + bt + c)e^{3t}$

$\dot{x}(t) = (2at + b)e^{3t} + 3(at^2 + bt + c)e^{3t}$

$= [3at^2 + (2a + 3b)t + (b + 3c)]e^{3t}$

$\ddot{x}(t) =$ etc. TEDIOUS

Is there an alternative?

SCENARIO: $\boxed{[p(D)]x(t) = e^{rt}u(t)}$

$D = \frac{d}{dt} \qquad p(D) = D^2 + 3D + 2I$

$D(e^{rt}u) = e^{rt}Du + re^{rt}u \qquad x(t) = e^{rt}u(t)$

$= e^{rt}[Du + ru]$

$= e^{rt}[(D + rI)u]$

$D^2(e^{rt}u) = D[D(e^{rt}u)] = D[e^{rt}(D + rI)u]$

$= e^{rt}[(D + rI)^2 u]$

etc: $D^n(e^{rt}u) = e^{rt}[(D + rI)^n u]$

$p(D)[e^{rt}u(t)] = e^{rt}[\underline{p(D + rI)\,u(t)}] = e^{rt}g(t)$

$$P(D+rI)\,u(t) = g(t)$$

Solve to get $u(t)$

Then $x(t) = e^{rt} u(t)$

VARIATION OF PARAMETERS II

SCENARIO: Solve $\ddot{x} + p_1(t)\dot{x} + p_0(t)x = R(t)$

FIRST SOLVE HOMOG EQN: $\{x_1(t), x_2(t)\}$

$$x_h(t) = C_1 x_1(t) + C_2 x_2(t).$$

SEEK PARTICULAR SOLUTION, but you're stuck.
VARY THE PARAMETERS.

SEEK $\quad x_p(t) = v_1(t)x_1(t) + v_2(t)x_2(t)$

$$x = v_1 x_1 + v_2 x_2$$

Product Rule $\quad \dot{x} = v_1\dot{x}_1 + v_2\dot{x}_2 + \cancel{\dot{v}_1 x_1 + \dot{v}_2 x_2}$

Set $\boxed{\dot{v}_1 x_1 + \dot{v}_2 x_2 = 0}$

$$\ddot{x} = v_1\ddot{x}_1 + v_2\ddot{x}_2 + \dot{v}_1\dot{x}_1 + \dot{v}_2\dot{x}_2$$

Subst into ODE:

$$\begin{aligned}
&\ddot{x} + p_1\dot{x} + p_0 x \\
&= v_1\ddot{x}_1 + v_2\ddot{x}_2 + \dot{v}_1\dot{x}_1 + \dot{v}_2\dot{x}_2 \\
&\quad + p_1(v_1\dot{x}_1 + v_2\dot{x}_2) + p_0(v_1 x_1 + v_2 x_2) \\
&= v_1[\cancel{\ddot{x}_1 + p_1\dot{x}_1 + p_0 x_1}] + v_2[\cancel{\ddot{x}_2 + p_1\dot{x}_2 + p_0 x_2}] \\
&\quad + \boxed{\dot{x}_1\dot{v}_1 + \dot{x}_2\dot{v}_2 = R(t)}
\end{aligned}$$

$$\left\{\begin{aligned} x_1\dot{v}_1 + x_2\dot{v}_2 &= 0 \\ \dot{x}_1\dot{v}_1 + \dot{x}_2\dot{v}_2 &= R(t)\end{aligned}\right\} \Rightarrow \begin{bmatrix} x_1 & x_2 \\ \dot{x}_1 & \dot{x}_2 \end{bmatrix}\begin{bmatrix}\dot{v}_1 \\ \dot{v}_2\end{bmatrix} = \begin{bmatrix} 0 \\ R(t)\end{bmatrix}$$

WRONSKIAN MATRIX $\rightarrow \dot{v}_1, \dot{v}_2$

Let $W(t) = \begin{bmatrix} x_1(t) & x_2(t) \\ \dot{x}_1(t) & \dot{x}_2(t) \end{bmatrix}$

$$W(t) \begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ R(t) \end{bmatrix}$$

If $\det[W(t)] \neq 0$ (WRONSKIAN CRITERION)

then $\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = [W(t)]^{-1} \begin{bmatrix} 0 \\ R(t) \end{bmatrix}$

$$[W(t)] = \frac{1}{\det[W(t)]} \begin{bmatrix} \dot{x}_2 & -x_2 \\ -\dot{x}_1 & x_1 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = \frac{1}{x_1\dot{x}_2 - x_2\dot{x}_1} \begin{bmatrix} \dot{x}_2 & -x_2 \\ -\dot{x}_1 & x_1 \end{bmatrix} \begin{bmatrix} 0 \\ R \end{bmatrix}$$

$$\boxed{\begin{aligned} \dot{v}_1 &= \frac{dv_1}{dt} = \frac{-Rx_2}{x_1\dot{x}_2 - x_2\dot{x}_1} \\ \dot{v}_2 &= \frac{dv_2}{dt} = \frac{+Rx_1}{x_1\dot{x}_2 - x_2\dot{x}_1} \end{aligned}}$$

Generalizes to higher order linear ODE's

3rd order $\longrightarrow$ $x_h(t) = \underline{\underline{c_1}}\, x_1(t) + \underline{\underline{c_2}}\, x_2(t) + \underline{\underline{c_3}}\, x_3(t)$

$$\Rightarrow \begin{bmatrix} x_1 & x_2 & x_3 \\ \dot{x}_1 & \dot{x}_2 & \dot{x}_3 \\ \ddot{x}_1 & \ddot{x}_2 & \ddot{x}_3 \end{bmatrix} \begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \\ \dot{v}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ R \end{bmatrix} \rightarrow \dot{v}_1, \dot{v}_2, \dot{v}_3$$

$\Rightarrow v_1 \quad v_2 \quad v_3$

$x_p = v_1 x_1 + v_2 x_2 + v_3 x_3$

Example: Find part. soln to

$$\ddot{x} + 3\dot{x} + 2x = e^{-t}$$

VAR'N of Params:

Homog $p(s) = s^2 + 3s + 2 = (s+2)(s+1) = 0$

$$s = -2,\ s = -1 \Rightarrow \boxed{x_h(t) = c_1 \underbrace{e^{-2t}}_{x_1} + c_2 \underbrace{e^{-t}}_{x_2}}$$

$$\begin{bmatrix} x_1 & x_2 \\ \dot{x}_1 & \dot{x}_2 \end{bmatrix} \begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ R \end{bmatrix} \qquad x_p = v_1 x_1 + v_2 x_2$$

$$\begin{bmatrix} e^{-2t} & e^{-t} \\ -2e^{-2t} & -e^{-t} \end{bmatrix} \begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ e^{-t} \end{bmatrix}$$

$$\frac{dv_1}{dt} = \frac{-e^{-t}e^{-t}}{e^{-3t}} = -\frac{e^{-2t}}{e^{-3t}} = -e^{t}$$

$$\frac{dv_2}{dt} = \frac{e^{-t}e^{-2t}}{e^{-3t}} = \frac{e^{-3t}}{e^{-3t}} = 1$$

$$v_1(t) = -e^{t} \qquad v_2(t) = t$$

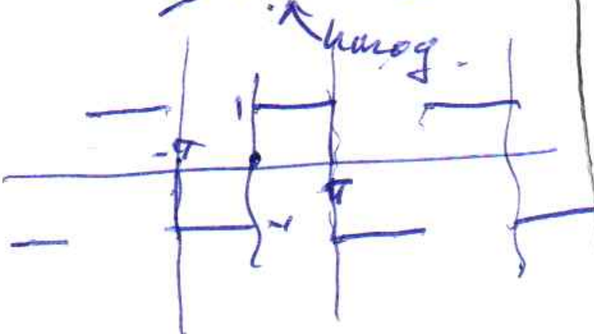
$$\therefore\ x_p(t) = v_1 x_1 + v_2 x_2$$

$$= -e^{t}e^{-2t} + te^{-t}$$

$$= \cancel{-e^{-t}} + \boxed{te^{-t}}$$

homog.

sq(t)



ALSO:
DISCONTINUOUS INPUTS
and
INTRO TO
Fourier Series