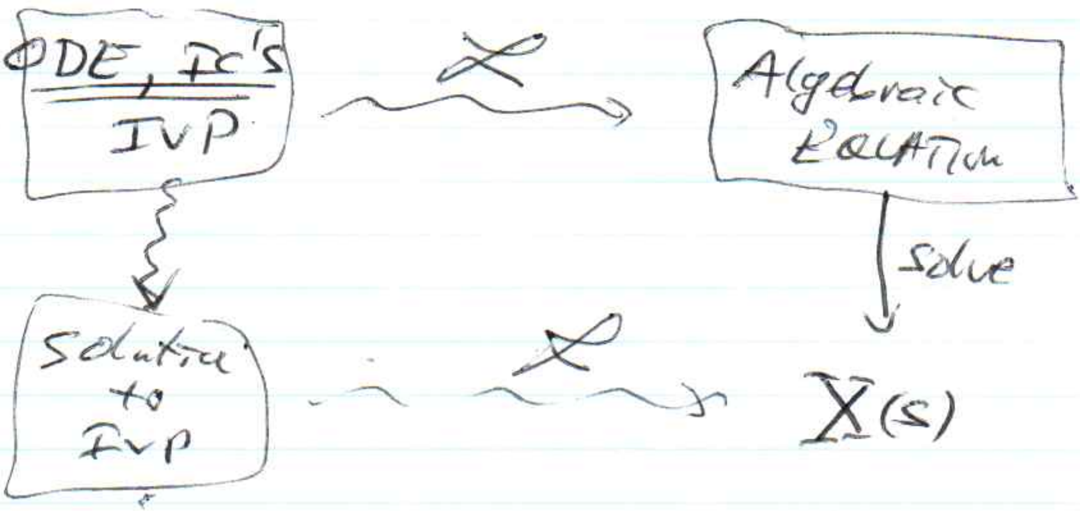
BIG IDEA : LAPLACE TRANSFORM

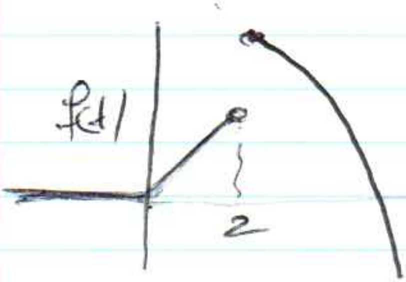


Heaviside $u(t)$, Delta function $\delta(t)$

Generalized Derivative $u'(t) = \delta(t)$

$$\int_{-\infty}^{+\infty} \delta(t)\,dt = 1 \qquad \int_{-\infty}^{+\infty} \delta(t-2)\,dt = 1$$

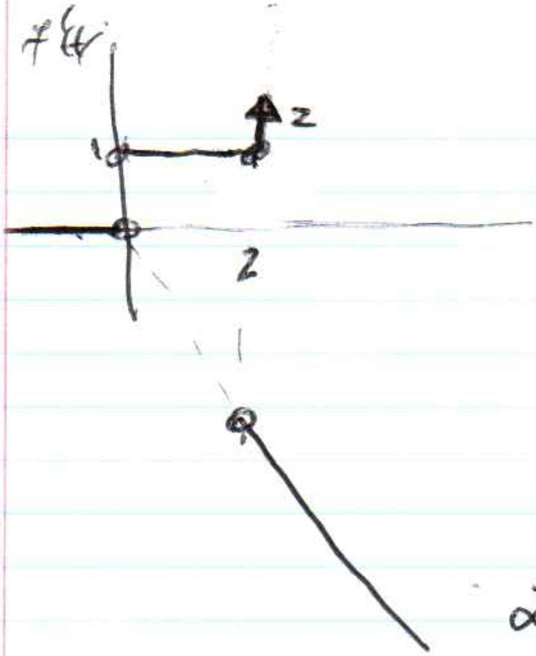
<u>Example:</u> $f(t) = \begin{cases} 0 & t \leq 0 \\ t & 0 < t \leq 2 \\ 8 - t^2 & t > 2 \end{cases}$



$$f(t) = \cancel{0} + t[u(t) - u(t-2)] + (8-t^2)[u(t-2)]$$

$$f'(t) = t[\delta(t) - \delta(t-2)] + [u(t) - u(t-2)] + (8-t^2)\,\delta(t-2) - 2t\,u(t-2)$$

$$= -2\delta(t-2) + [u(t) - u(t-2)] + 4\delta(t-2) - 2t\,u(t-2)$$



Laplace Transform

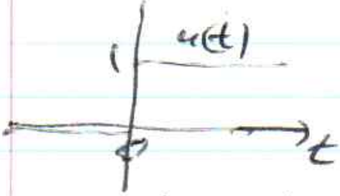
$$f(t) \xrightarrow{\mathcal{L}} F(s)$$

$$\mathcal{L}[f(t)] = F(s) = \int_{0^-}^{+\infty} e^{-st} f(t)\,dt$$

$$\mathcal{L}(x(t)) = X(s) = \int_{0^-} e^{-st} x(t)\,dt$$

Showed: $\mathcal{L}$ is linear

$$\mathcal{L}(1) = \mathcal{L}[u(t)] = \frac{1}{s}$$



$$\mathcal{L}[t] = \frac{1}{s^2}$$

$t\,u(t)$

$$\mathcal{L}[t^2] = \frac{2}{s^3}$$

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}[\text{polynomial}]$$

s-Derivative Rule

$$\mathcal{L}[f(t)] = F(s) = \int_{0^-}^{\infty} e^{-st} f(t)\,dt$$

$$\mathcal{L}[t\,f(t)] = -F'(s)$$

$$\mathcal{L}[t\,f(t)] = \int_{0^-}^{+\infty} e^{-st}\, t\, f(t)\,dt$$

$$F'(s) = \frac{d}{ds}\int_{0^-}^{+\infty} e^{-st} f(t)\,dt = \boxed{-\int_{0^-}^{+\infty} e^{-st}\, t\, f(t)\,dt}$$

$$\therefore \mathcal{L}[t\,f(t)] = -F'(s)$$

Transform exponential function.

$$\mathcal{L}[e^{at}] = \mathcal{L}[e^{at}u(t)] = \int_{0^-}^{+\infty} e^{-st}e^{at}\,dt$$

$$= \int_{0^-}^{+\infty} e^{-(s-a)t}\,dt = \frac{1}{s-a}$$

By s-deriv Rule

$$\mathcal{L}[t e^{at}] = -\left[\frac{-1}{(s-a)^2}\right] = \frac{+1}{(s-a)^2}$$

<u>S-shift Rule</u>: $\mathcal{L}[e^{rt}f(t)] = F(s-r)$

$$\mathcal{L}[f(t)] = F(s)$$

<u>Why?</u> $\mathcal{L}[e^{rt}f(t)] = \int_{0^-}^{+\infty} e^{-st}e^{rt}f(t)\,dt$

$$= \int_{0^-}^{+\infty} e^{-(s-r)t}f(t)\,dt = F(s-r)$$

<u>Delta Function</u>

$$\mathcal{L}[\delta(t)] = \int_{0^-}^{+\infty} e^{-st}\delta(t)\,dt = 1$$

<u>General FACT</u>: $\int_{-\infty}^{+\infty} f(t)\delta(t)\,dt = f(0)$

$$\int_{-\infty}^{+\infty} f(t)\delta(t-a)\,dt = f(a)$$

$$e^{i\omega t} = \cos\omega t + i\sin\omega t$$

$$\mathcal{L}(e^{i\omega t}) = \mathcal{L}(\cos\omega t) + i\,\mathcal{L}(\sin\omega t)$$

LAST TIME

$$\mathcal{L}[\cos \omega t] = \frac{s}{s^2+\omega^2}$$

$$\mathcal{L}[\sin \omega t] = \frac{\omega}{s^2+\omega^2}$$

Inverse Laplace TRANSFORM

Suppose

$$\mathcal{L}(x(t)) = X(s) = \frac{3}{s^2} + \frac{4s}{s^2+9} + \frac{2}{3}\left[\frac{2 \cdot 3}{s^2+9}\right]$$

then $x(t) = 3t + 4\cos 3t + \frac{2}{3}\sin 3t$

<u>TRANSFORMING DERIVATIVES</u>

Suppose $\mathcal{L}[f(t)] = F(s)$

$$\mathcal{L}[f'(t)] = \int_{0^-}^{+\infty} e^{-st} f'(t)\,dt$$

$$u = e^{-st} \quad dv = f'(t)\,dt$$

$$du = -se^{-st}\,dt \quad v = f(t)$$

$$\therefore = \left. e^{-st} f(t) \right]_{0^-}^{+\infty} + s\underbrace{\int_{0^-}^{+\infty} e^{-st} f(t)\,dt}_{F(s)}$$

$$0 - f(0) + sF(s)$$

$$\mathcal{L}[f'(t)] = sF(s) - f(0)$$

$$\mathcal{L}[f''(t)] = s\,\mathcal{L}[f'(t)] - f'(0)$$

$$= s[sF(s) - f(0)] - f'(0)$$

$$= s^2F(s) - s \cdot f(0) - f'(0)$$

$$\mathcal{L}[f'''(t)] = s\,\mathcal{L}[f''(t)] - f''(0)$$

$$= s\left[s^2 F(s) - s f(0) - f'(0)\right] - f''(0)$$

$$= s^3 F(s) - s^2 f(0) - s f'(0) - f''(0)$$

$$\mathcal{L}[f^{(n)}(t)] = s^n F(s) - s^{n-1} f(0) - \cdots - s f - \cdots$$

In an ODE

$$\mathcal{L}[\dddot{x}(t)] = s^3 X(s) - s^2 x(0) - s\dot{x}(0) - \ddot{x}(0)$$

<u>If</u> $\boxed{\text{Rest Initial Conditions}}$

$$\mathcal{L}[\dddot{x}(t)] = s^3 X(s)$$

<u>ODE:</u> $[p(D)]\,x(t) = g(t)$

$$\downarrow \mathcal{L}$$

$$p(s)\,X(s) = G(s)$$

$$\boxed{X(s) = \frac{G(s)}{p(s)}}$$ only if we have Rest Initial Conditions

<u>UNIT IMPULSE RESPONSE</u>

Solve $[p(D)]\,x(t) = \delta(t)$ $\boxed{\text{REST INITIAL CONTIONS}}$

$$\downarrow \mathcal{L}$$

call solution $w(t)$

$$p(s)\,X(s) = 1$$

$$X(s) = \frac{1}{p(s)}$$

<u>weight function</u>

$\mathcal{L}[w(t)] = W(s) =$ TRANSFER Function

UNIT STEP RESPONSE

$[p(D)]\,x(t) = u(t)$

$\mathcal{L} \downarrow$ [REST INITIAL CONDITIONS]

$[p(s)]\,X(s) = \frac{1}{s}$

$X(s) = \frac{1}{s\,p(s)}$

<u>Note:</u> $\mathcal{L}[\dot{u}(t)] = s\,\mathcal{L}[u(t)]$

$\mathcal{L}[\delta(t)] = s \cdot \frac{1}{s} = 1$

<u>Examples</u> $p(D) = D + 3I$ unit Impulse + unit step Response

① $\dot{x} + 3x = \delta(t)$, $x(0)$

$sX + 3X = 1$

$(s+3)X(s) = 1$ $p(s) = s+3$

$X(s) = \frac{1}{s+3}$

Recognize that $x(t) = e^{-3t} = u(t)$

② $\dot{x} + 3x = u(t)$, $x(0) = 0$

$\downarrow$

$(s+3)X(s) = \frac{1}{s}$

$X(s) = \frac{1}{s(s+3)} = \frac{A}{s} + \frac{B}{s+3} = \frac{1}{3}\left(\underset{1}{\frac{1}{s}} - \underset{e^{-3t}}{\frac{1}{s+3}}\right)$

$1 = A(s+3) + B(s)$ $s = -3$ $-3B = 1$

$s = 0$ $3A = 1$ $A = \frac{1}{3}$ $B = -\frac{1}{3}$

$x(t) = \frac{1}{3}(1 - e^{-3t}) =$

HARMONIC RESPONSE

$$\boxed{\ddot{x} + \omega^2 x = \delta(t)}$$ Rest IC's

$x(0)=0, \ \dot{x}(0)=0$

$p(D) = D^2 + \omega^2 I$

$p(s) = s^2 + \omega^2$

unit Impulse Response

$\mathcal{L}[\sin(\omega t)]$

$$\mathcal{L} \rightarrow (s^2+\omega^2) X(s) = 1$$

$$X(s) = \frac{1}{s^2+\omega^2} = \frac{1}{\omega}\left(\frac{\omega}{s^2+\omega^2}\right)$$

$$\downarrow$$

$$x(t) = \frac{1}{\omega} \sin \omega t$$

LAPLACE DIRECT

✓ non-Rest

Solve $\dot{x} + 3x = 3\cos 2t \quad x(0) = 2$

$$\downarrow \mathcal{L}$$

$$[sX - x(0)] + 3X = 3 \cdot \frac{s}{s^2+4}$$

$$(s+3)X - 2 = \frac{3s}{s^2+4}$$

$$(s+3)X = 2 + \frac{3s}{s^2+4} = \frac{2s^2+3s+8}{s^2+4}$$

$$X(s) = \frac{2s^2+3s+8}{(s+3)(s^2+4)}$$ PARTIAL FRACTION

$$= \frac{A}{s+3} + \frac{Bs+C}{s^2+4}$$

$$2s^2+3s+8 = A(s^2+4) + (Bs+C)(s+3)$$

$$s=-3 \quad 17 = 13A \quad A = \frac{17}{13}$$

$$s = 2i$$

$$\cancel{-8}+6i\cancel{+8} = 6i = (2Bi+C)(3+2i)$$

$$6i = (3C-4B) + i(6B+2C)$$

$$\begin{aligned} 3C-4B &= 0 \\ 6B+2C &= 6 \end{aligned} \quad \rightarrow B, C$$

<u>OR</u>

$$2s^2+3s+8 = (A+B)s^2 + (3B+C)s + (4A+3C)$$

$$\begin{aligned} A+B &= 2 \\ 3B+C &= 3 \\ 4A+3C &= 8 \end{aligned} \quad \rightarrow \quad A = \frac{17}{13} \quad B = \frac{9}{13} \quad C = \frac{12}{13}$$

$$X(s) = \frac{1}{13}\left[\frac{17}{s+3} + \frac{9s}{s^2+4} + \frac{12}{s^2+4}\right]$$

$$6\left(\frac{2}{s^2+4}\right)$$

$$x(t) = \frac{1}{13}\left[17e^{-3t} + 9\cos 2t + 6\sin 2t\right]$$

Solve $\ddot{x} + 3\dot{x} + 2x = 4 \qquad x(0) = \dot{x}(0) = 0$

Prev. methods $p(s) = s^2 + 3s + 2$

$= (s+2)(s+1) = 0$

$s = -2, \; s = -1$

$x(t) = c_1 e^{-2t} + c_2 e^{-t} + 2$

$\dot{x}(t) = -2c_1 e^{-2t} - c_2 e^{-t}$

$\cancel{x(0) = c_1 + c_2 = 0}$

$x_p(t) = 2$

$x(0) = c_1 + c_2 + 2 = 0$

$\dot{x}(0) = -2c_1 - c_2 = 0$

$c_1 + c_2 = -2 \qquad -c_1 = -2$

$-2c_1 - c_2 = 0 \qquad c_1 = 2$

$c_2 = -4$

$x(t) = 2e^{-2t} - 4e^{-t} + 2$

$\mathcal{L} \quad (s^2 + 3s + 2) X(s) = \frac{4}{s}$

$X(s) = \frac{4}{s(s+2)(s+1)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+1}$

$4 = A(s+2)(s+1) + Bs(s+1) + Cs(s+2)$

$s = 0 \quad 4 = 2A \quad A = 2$

$s = -1 \quad 4 = -C \quad C = -4$

$s = -2 \quad 4 = 2B \quad B = 2$

$x(t) = A \cdot 1 + Be^{-2t} + Ce^{-t}$

$= 2 + 2e^{-2t} - 4e^{-t}$