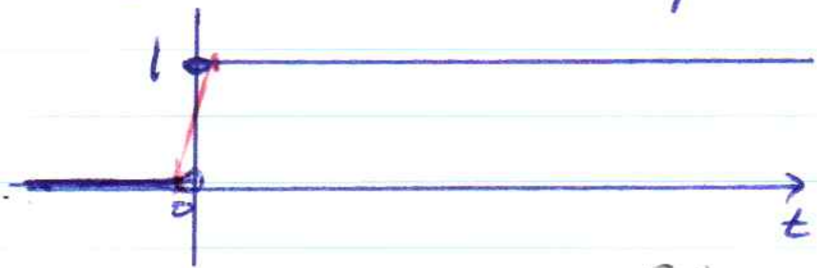
Generalized Calculus

Heaviside function (Step function)



$$u(t) = \begin{Bmatrix} 0 & t<0 \\ 1 & t\geq 0 \end{Bmatrix}$$

$$\delta(t) = u'(t) = \begin{cases} 0 & t<0 \\ \infty & t=0 \\ 0 & t>0 \end{cases}$$

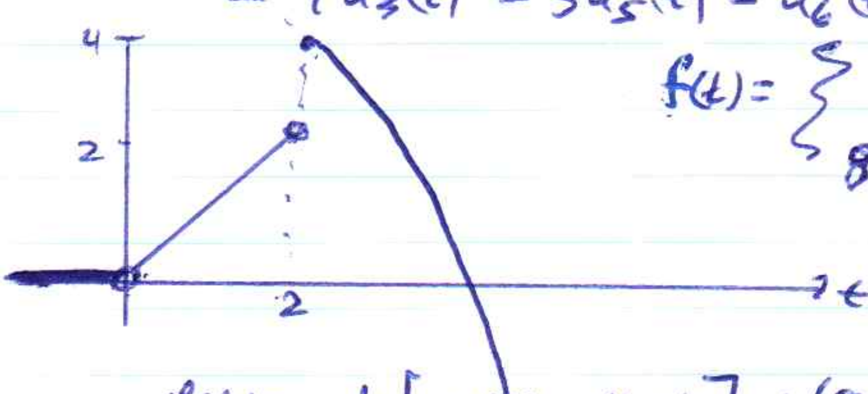
$$u_a(t) = u(t-a) = \begin{Bmatrix} 0 & t<a \\ 1 & t\geq a \end{Bmatrix}$$

Box Functions (switch on - switch off)

$$g(t) = \begin{Bmatrix} 0 & t<3 \\ 4 & 3<t<5 \\ 1 & 5<t<6 \\ 0 & t>6 \end{Bmatrix}$$

$$g(t) = 4[u_3(t) - u_5(t)] + 1[u_5(t) - u_6(t)]$$

$$= 4u_3(t) - 3u_5(t) - u_6(t)$$



$$f(t) = \begin{Bmatrix} 0 & t<0 \\ t & 0<t<2 \\ 8-t^2 & t>2 \end{Bmatrix}$$

$$f(t) = t[u(t) - u_2(t)] + (8-t^2)\,u_2(t)$$

DERIVATIVE OF HEAVISIDE FUNCTION

Delta Function $u'(t) = \delta(t) = \begin{cases} 0 & t<0 \\ \infty & t=0 \\ 0 & t>0 \end{cases}$

$$\int_{-\infty}^{+\infty} \delta(t)\,dt = 1$$

$u_a(t) = u(t-a)$ $\quad u_a'(t) = \begin{cases} 0 & t<a \\ \infty & t=a \\ 0 & t>a \end{cases}$

$$\int_{-\infty}^{+\infty} u_a'(t)\,dt$$

$u_a'(t) = \delta_a(t)$
$= \delta(t-a)$

$$= \int_{-\infty}^{+\infty} \delta_a(t)\,dt = 1$$

BIG IDEA $\displaystyle\int_{-\infty}^{+\infty} f(t)\,\delta(t)\,dt = f(0)$

$$\int_{-\infty}^{+\infty} f(t)\,\delta_a(t)\,dt = f(a)$$

Evaluation of a function is the same as "integration against a delta function".

LINEAR FUNCTIONALS and measurement

$\boxed{\mathbb{R}^n}$ $\quad \overrightarrow{\text{vector}} \longrightarrow \text{number}$

$\vec{v} \in \mathbb{R}^3 \quad \vec{v} = v_x \vec{i} + v_y \vec{j} + v_z \vec{k}$

$\text{comp}_x(\vec{v}) = v_x$
$\text{comp}_y(\vec{v}) = v_y$
$\text{comp}_z(\vec{v}) = v_z$

Functionals   function $\longrightarrow$ number

Fourier coefficients $f(t)$

$$f(t) \longrightarrow a_0 = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(t)\,dt$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(t)\cos nt\,dt$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(t)\sin nt\,dt$$

Evaluation just a linear functional.

$$f \longrightarrow f(0)$$

$$f \longrightarrow f(a)$$

IDEA: Generalized Derivative.

[BASED on Integration by Parts]

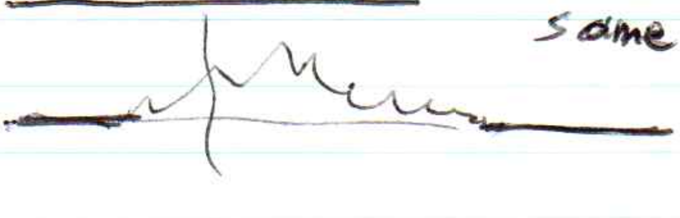
Product Rule: $\frac{d}{dt}\left[uv\right] = u\frac{dv}{dt} + v\frac{du}{dt}$

$$\left[uv\right]_a^b = \int_a^b \frac{d}{dt}\left[uv\right]dt = \int_a^b u\frac{dv}{dt}\,dt + \int_a^b v\frac{du}{dt}\,dt$$

$$\left[u(t)\,v(t)\right]_{t=a}^{t=b} = \int_a^b u(t)\,dv + \int_a^b v\,du$$

$$\left[\int u\,dv = uv - \int v\,du\right]$$

COMPACT SUPPORT : VANISHES outside some finite interval



$g(t)$ with compact support

$$\int_a^b f'(t)g(t)dt = g(t)f(t)\Big|_a^b - \int_a^b f(t)g'(t)dt$$

$$\int_{-\infty}^{+\infty} f'(t)g(t)dt = 0 - \int_{-\infty}^{+\infty} f(t)g'(t)dt$$

"A function is only as good as how it integrates against other functions."

DERIVATIVE OF HEAVISIDE FUNCTION

$$f(t) = u(t) \qquad u(t) = \begin{cases} 0 & t<0 \\ 1 & t>0 \end{cases}$$

$$f'(t) = u'(t)$$

$$\int_{-\infty}^{+\infty} u'(t)g(t)dt = -\int_{-\infty}^{+\infty} u(t)g'(t)dt$$

$\boxed{\text{ETC}}$

$$= -\int_0^{+\infty} g'(t)dt = -g(t)\Big|_0^{+\infty}$$

$$= -\left[0 - g(0)\right] = g(0)$$

$\boxed{\text{Generalized Calculus}}$

$$u'(t) = \delta(t)$$

$$\int_{-\infty}^{+\infty} g(t)\delta(t)dt = g(0)$$

Special Case $\int_{-\infty}^{+\infty} \delta(t) = 1$

$$\int_{-\infty}^{+\infty} \delta(t)dt = \int_{-\infty}^{+\infty} u'(t)dt = u(t)\Big|_{-\infty}^{\infty}$$

$$= 1 - 0 = 1$$ "probability density"

LAPLACE TRANSFORM

BIG IDEA

$$\begin{tikzcd}[column sep=huge, row sep=large]
\boxed{\text{LINEAR ODE w/ IC's}} \arrow[r, "\mathcal{L}"] \arrow[d, "\text{Solve}"] & \boxed{\text{Algebraic EQN}} \arrow[d, "\text{Solve via algebra}"] \\
\boxed{\text{Response (Soln To IVP)}} \arrow[r, shift left, "\mathcal{L}"] & \boxed{\text{Solution}} \arrow[l, shift left, "\mathcal{L}^{-1}"]
\end{tikzcd}$$

$$\mathcal{L}\left[f(t)\right] = F(s) = \int_{0^-}^{+\infty} \underbrace{e^{-st}}_{\text{Integral Kernel}} f(t)\,dt$$

$\mathcal{L}$ is Linear

$$\mathcal{L}\left[a f(t) + b g(t)\right] = \int_{0^-}^{+\infty} e^{-st}\left(a f(t) + b g(t)\right) dt$$

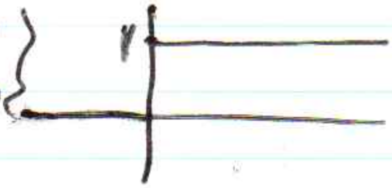
$$= a \int_{0^-}^{+\infty} e^{-st} f(t) + b \int_{0^-}^{+\infty} e^{-st} g(t)\,dt$$

$$= a\,\mathcal{L}\left[f(t)\right] + b\,\mathcal{L}\left[g(t)\right]$$

$$= a\,F(s) + b\,G(s)$$

BASIC TRANSFORMS   only care about $t > 0$

$f(t) = 1 \qquad u(t) = \begin{cases} 0 & t<0 \\ 1 & t>0 \end{cases}$



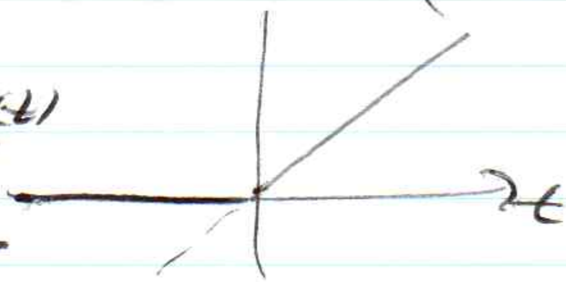
① $\mathcal{L}(1) = \mathcal{L}[u(t)]$

$$= \int_{0^-}^{+\infty} e^{-st}\, u(t)\, dt = \int_{0^-}^{+\infty} e^{-st}\, 1\, dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^{+\infty} = 0 + \frac{1}{s} = \frac{1}{s}$$

$$\mathcal{L}[1] = \frac{1}{s} \qquad \mathcal{L}[c] = \frac{c}{s} \ \left(\text{by linearity}\right)$$

② $f(t) = t \longleftrightarrow t\,u(t)$



$$\mathcal{L}[t] = \int_{0^-}^{+\infty} t\, e^{-st}\, dt$$

Integ. by parts $\quad u = t \quad dv = e^{-st} dt$

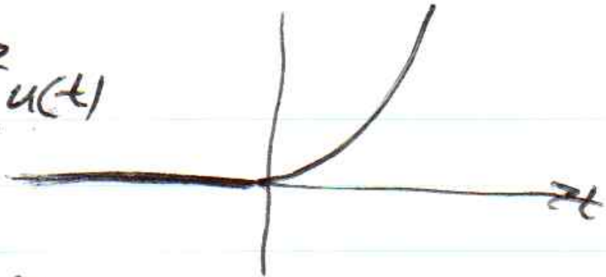
$du = dt \quad v = \frac{e^{-st}}{-s}$

$$\mathcal{L}[t] = \int_{0^-}^{+\infty} t e^{-st} dt = \cancel{\left.\frac{te^{-st}}{-s}\right|_{0^-}^{+\infty}} + \frac{1}{s}\boxed{\int_{0^-}^{+\infty} e^{-st} dt}$$

$$= \frac{1}{s}\frac{1}{s} = \frac{1}{s^2}$$

$$\mathcal{L}[t] = \frac{1}{s^2}$$

$f(t) = t^2 \qquad t^2 u(t)$



$$\mathcal{L}[t^2] = \int_{0^-}^{+\infty} e^{-st} t^2 \, dt$$

PARTS $\quad u = t^2 \quad dv = e^{-st}\,dt$

$\quad du = 2t\,dt \quad v = \frac{e^{-st}}{-s}$

$$= \cancel{\frac{t^2 e^{-st}}{-s}\Big|_{0^-}^{+\infty}} + \frac{2}{s}\boxed{\int_{0^-}^{+\infty} t e^{-st}\, dt}$$

$$= \frac{2}{s}\left(\frac{1}{s^2}\right) = \frac{2}{s^3}$$

$\mathcal{L}[1] = \frac{1}{s}$

$\mathcal{L}[t] = \frac{1}{s^2}$

$\mathcal{L}[t^2] = \frac{2}{s^3} = \frac{2!}{s^3}$

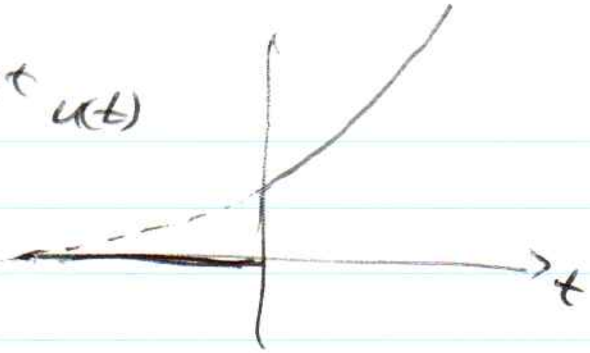
$\mathcal{L}[t^3] = \frac{3!}{s^4}$

$\vdots$

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$$

$\mathcal{L}$[polynomial]

$\mathcal{L}(e^{at})$ $e^{at}u(t)$



$\|$

$$\int_{0^-}^{+\infty} e^{-st} e^{at}\,dt$$

$$= \int_{0^-}^{+\infty} e^{-(s-a)t}\,dt = \left.\frac{e^{-(s-a)t}}{-(s-a)}\right|_{0}^{+\infty}$$

$$= +\frac{1}{s-a}$$

$$\mathcal{L}(e^{at}) = \frac{1}{s-a}$$

$\boxed{\delta(t)}$ "generalized function".

$$\mathcal{L}[\delta(t)] = \int_{0^-}^{+\infty} e^{-st}\delta(t)\,dt$$

$$= \int_{-\infty}^{+\infty} e^{-st}\delta(t)\,dt = 1$$

$$\mathcal{L}[\delta(t)] = 1$$

$\mathcal{L}[\cos\omega t]$ $\mathcal{L}[\sin\omega t]$

Euler's Identity

$$e^{i\omega t} = \cos\omega t + i\sin\omega t$$

$$\mathcal{L}\left[e^{i\omega t}\right] = \mathcal{L}(\cos\omega t) + i\,\mathcal{L}(\sin\omega t)$$

$$\parallel$$

$$\frac{1}{s-i\omega}\left[\frac{s+i\omega}{s+i\omega}\right] = \frac{s+i\omega}{s^2+\omega^2}$$

$$= \left(\frac{s}{s^2+\omega^2}\right) + i\left(\frac{\omega}{s^2+\omega^2}\right)$$

$$\mathcal{L}(\cos\omega t) = \frac{s}{s^2+\omega^2} \quad \text{(Real part)}$$

$$\mathcal{L}(\sin\omega t) = \frac{\omega}{s^2+\omega^2} \quad \text{(Imag. part)}$$