PERIODIC INPUTS + Fourier Series

$$\frac{d^n x}{dt^n} + p_{n-1}(t)\frac{d^{n-1}x}{dt^{n-1}} + \cdots + p_1(t)\frac{dx}{dt} + p_0(t)x = \underline{\underline{f(t)}}$$

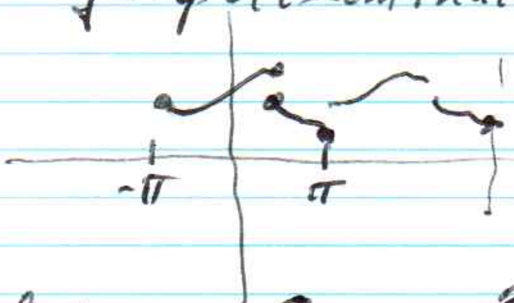
Input $f(t)$ periodic $f(t+nT) = f(t)$

Specialize to standard $[-\pi, \pi]$

period $= 2\pi$

constant, $\cos t$, $\sin t$, $\cos 2t$, $\sin 2t$, etc.

<u>Thm [Fourier]</u>: If $f(t)$ is periodic with base period 2π, continuous except at a finite no. of jump discontinuities, then:



$f(t)$ can be represented by a convergent Fourier series of the form

$$f(t) \sim \frac{a_0}{2} + \sum_{n\geq 1}^{\infty}\left(a_n \cos nt + b_n \sin nt\right)$$

where the Fourier coefficients are calculated by:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(t)\,dt, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(t)\cos nt\,dt$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(t)\sin nt\,dt$$

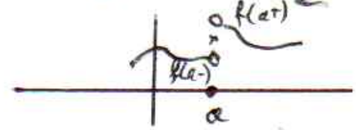
<u>Even</u>: $f(-t) = f(t)$

$b_n = 0$

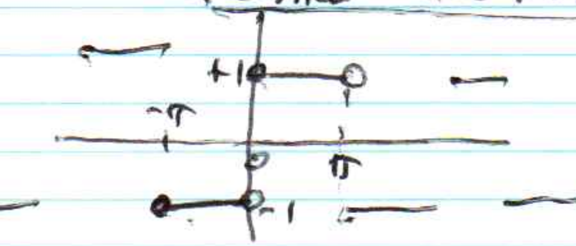
<u>ODD</u>: $f(-t) = -f(t)$

$a_n = 0$

Also: Fourier Series converges to value of $f(t)$ at any point of continuity, and to the average $\dfrac{f(a-)+f(a+)}{2}$ at any jump discontinuity.



SQUARE WAVE FUNCTION



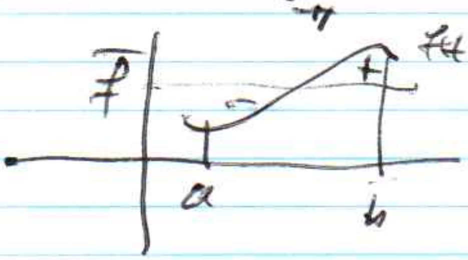
$$sq(t) = \begin{cases} -1 & [-\pi, 0) \\ +1 & [0, +\pi) \end{cases}$$

ODD Function

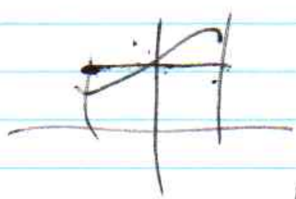
$a_0 = 0$, all $a_n = 0$

Note: constant term: $\dfrac{a_0}{2} = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{+\pi} f(t)\,dt = \bar{f}$ (average value on the interval)

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(t)\,dt$$



$$\bar{f} = \frac{\int_a^b f(t)\,dt}{b-a}$$



$$\left[\begin{array}{c}\cos n\pi = (-1)^n \\ \text{integers } n\end{array}\right]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(t)\sin nt\,dt = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\sin nt\,dt + \int_0^{\pi} \sin nt\,dt\right]$$

$$\int \sin nt\,dt = \frac{-\cos nt}{n}$$

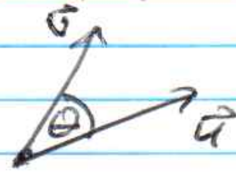
$$\Rightarrow \frac{1}{n\pi}\left[+ \left[\frac{+\cos nt}{\cancel{n}}\right]_{-\pi}^{0} - \left[\frac{+\cos nt}{\cancel{n}}\right]_{0}^{\pi}\right]$$

$$= \frac{1}{n\pi}\left[\left[1 - (-1)^n\right] - \left[(-1)^n - 1\right]\right]$$

$$= \frac{1}{n\pi}\left[2 - 2(-1)^n\right] = \frac{2}{n\pi}\left[1 - (-1)^n\right]$$

$$= \begin{cases} 0 & n \text{ even} \\ \dfrac{4}{n\pi} & n \text{ odd} \end{cases} \Rightarrow \boxed{\sum_{n \text{ odd}} \frac{4}{n\pi}\sin nt}$$

DOT PRODUCT in $\mathbb{R}^n$



$$\vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta$$

$$l = \vec{V}\cdot\frac{\vec{u}}{\|\vec{u}\|}$$ Scalar projection.

unit vector

l = scalar projection

$$\cos\theta = \frac{l}{\|\vec{v}\|}$$

$$l = \|\vec{v}\|\cos\theta$$

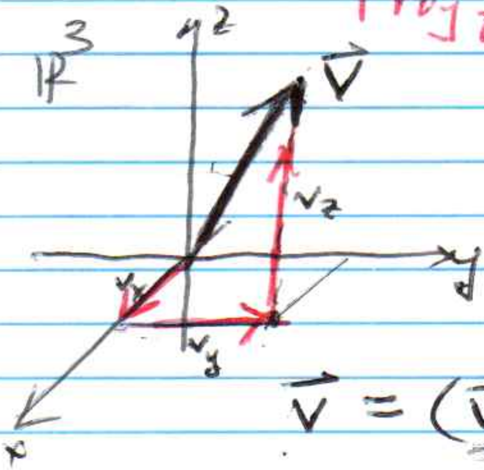
If $\vec{u}$ is a unit vector

$$l = \vec{V}\cdot\vec{u}$$
$$= \vec{v}\cdot\begin{pmatrix}\text{unit vector in} \\ \text{desired direction}\end{pmatrix}$$

vector projection.

$$\text{Proj}_{\vec{u}}\vec{V} = (\vec{v}\cdot\vec{u})\,\vec{u}$$

$\mathbb{R}^3$



$$v_x = \vec{V}\cdot\vec{i}$$
$$v_y = \vec{V}\cdot\vec{j}$$
$$v_z = \vec{v}\cdot\vec{k}$$

$$\vec{V} = (\vec{v}\cdot\vec{i})\,\vec{i} + (\vec{v}\cdot\vec{j})\,\vec{j} + (\vec{v}\cdot\vec{k})\,\vec{k}$$

Dot product $\vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n$

$$= u(1)\,v(1) + u(2)\,v(2) + \cdots + u(n)\,v(n)$$

Let's do this with "continuous" function on $[\pi, +\pi]$

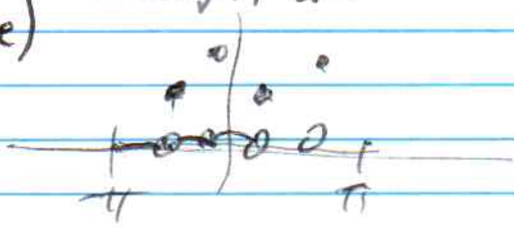
f, g piecewise continuous on $[-\pi, +\pi]$

$$\langle f, g\rangle = K\int_{-\pi}^{+\pi} f(t)g(t)\,dt \qquad K>0$$

choose $K = \frac{1}{\pi}$

$$\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(t)g(t)\,dt \quad \left(\begin{array}{c}\text{Inner}\\ \text{Product}\end{array}\right)$$

DOT PRODUCT	Inner Product
① $\vec{v}\cdot\vec{u} = \vec{u}\cdot\vec{v}$	① $\langle g, f\rangle = \langle f, g\rangle$
② $\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w}$ $(\vec{u}+\vec{v})\cdot\vec{w} = \vec{u}\cdot\vec{w} + \vec{v}\cdot\vec{w}$	② $\langle f, g+h\rangle = \langle f, g\rangle + \langle f, h\rangle$ $\langle f+g, h\rangle = \langle f, h\rangle + \langle g, h\rangle$
③ $\vec{u}\cdot(t\vec{v}) = t(\vec{u}\cdot\vec{v})$ $= (t\vec{u})\cdot\vec{v}$	③ $\langle cf, g\rangle = c\langle f, g\rangle$ $= \langle f, cg\rangle$
④ $\vec{u}\cdot\vec{u} = \lVert\vec{u}\rVert^2 \geq 0$ (positive definite) $= 0$ only if $\vec{u} = \vec{0}$	④ $\langle f, f\rangle = \lVert f\rVert^2 \geq 0$ norm $= 0$ only if $f = 0$ "almost everywhere" (not quite positive definite)



$\mathbb{R}^n$: ON BASIS. mutually orthogonal unit vectors. [ORTHONORMAL BASIS]

$\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$

ON BASIS

$$\vec{V} = V_x\vec{\imath} + V_y\vec{\jmath} + V_z\vec{k}$$

Refer to lecture #7 Notes

With functions:

$$\left\{\frac{1}{\sqrt{2}}, \cos t, \sin t, \cos 2t, \sin 2t, \ldots, \cos nt, \sin nt\right\}$$

$2n+1$ function.

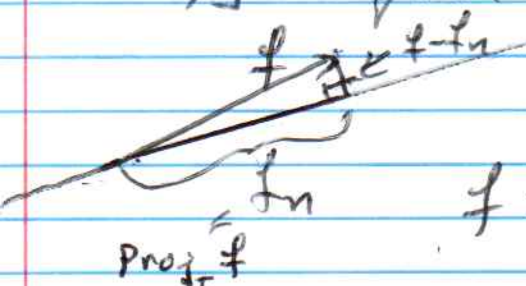
$\Rightarrow$ ON (orthonormal) BASIS [for a $2n+1$ dimensional subspace of functions.]

all have norm $= 1$

all mutually orthogonal.

$$\vec{u} \perp \vec{v} \Leftrightarrow \vec{u} \cdot \vec{v} = 0$$

f, g orthogonal iff $\langle f, g \rangle = 0$



$$T_n = \text{Span}\left\{\frac{1}{\sqrt{2}}, \ldots \sin nt\right\}$$

$$f = \underbrace{(f - f_n)}_{\substack{\text{Remainder} \\ \text{(error)}}} + \underbrace{f_n}_{\in T_n}$$

Fourier: $\lim\limits_{n \to \infty} \|f - f_n\| = 0$

$$\|f\|^2 = \|f_n\|^2 + \|f - f_n\|^2 \quad \text{Pythagorean theorem}$$

$$\Rightarrow \lim_{n \to \infty} \|f_n\|^2 = \|f\|^2 \quad \rightarrow \text{Interesting FACTS}$$

$sq(t)$: $f(t) \sim \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\sin nt}{n}$ $\qquad b_n = \begin{cases} \frac{4}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$

$$\cancel{\frac{a_0}{2} + a_1^2 + b_1^2 + a_2^2 + b_2^2 + \cdots} = \|f\|^2$$

$$\left(\frac{4}{\pi}\right)^2 + \left(\frac{4}{3\pi}\right)^2 + \cdots = \|f\|^2$$

$$\|f\|^2 = \frac{1}{\pi}\int_{-\pi}^{+\pi} (f(t))^2\,dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} dt = \frac{1}{\pi}(2\pi) = 2$$

$$\left(\frac{4}{\pi}\right)^2 \Big[\underbrace{1 + \tfrac{1}{3^2} + \tfrac{1}{5^2} + \cdots}\Big] = 2$$

$$\sum_{n \text{ odd}} \frac{1}{n^2} = \frac{2\pi^2}{16} = \frac{\pi^2}{8}$$

SAWTOOTH

$f(t) = t$ on $[-\pi, +\pi]$

See Calculations in Lecture #7 notes.
Note especially use of Integration by Parts.

Also: Harmonic Response, Application to Linear ODE's with periodic Inputs, term-by-term solution using complex replacement and Exponential Response Formula

If Resonance, need to use Resonant Response Formula for a single term responsible for the resonance.
See Lecture Notes #7 for details.

Also, manipulations of Fourier Series using Integration, Differentiation, and algebraic manipulation.