Lecture #5 Zoom Notes

[most of class was done with Printed Lecture Notes]

Independence and adequacy of solns.

WRONSKIAN CRITERION

[Lecture #4 Pgs 6-7]

$$\underbrace{\ddot{x} + 3\dot{x} + 2x}_{} = f(t). \qquad x(t_0) = x_0$$

$$\dot{x}(t_0) = \dot{x}_0$$

$$p(s) = s^2 + 3s + 2$$

$$= (s+2)(s+1) = 0 \Rightarrow s = -2, s = -1.$$

$$\{f_1, f_2\} = \{e^{-2t}, e^{-t}\}$$

$$x_h(t) = c_1 e^{-2t} + c_2 e^{-t}$$

$$x_p(t) \qquad x(t) = x_h(t) + x_p(t)$$

$$\begin{bmatrix} f_1 & f_2 \\ f_1' & f_2' \end{bmatrix} \quad \det\begin{bmatrix} e^{-2t} & e^{-t} \\ -2e^{-2t} & -e^{-t} \end{bmatrix}$$

$$= -e^{-3t} + 2e^{-3t} = e^{-3t} \neq 0$$

ERF, RRF

ZIR + ZSR : Lect #5, pgs 3-4

ZERO INPUT RESPONSE

ZERO STATE RESPONSE

$$[p(D)]\, x(t) = a e^{rt}$$

Know how to find all homog. solns.

PARTICULAR SOLUTIONS

$$\text{ERF} \quad \boxed{x_p(t) = \frac{a e^{rt}}{p(r)}} \quad p(r) \neq 0$$

r not a characteristic ROOT

MASS-SPRING DASHPOT (MECHANICS)
LRC Circuits (electronics) } Analogy

Lecture #5, pages 1-3

DAMPED, overdamped, underdamped, CRITICALLY DAMPED*

MASS-SPRING DASHPOT system – decay modes

BIFURCATION LOCUS

* Repeated characteristic roots

Example: $\ddot{x} + 3\dot{x} + 2x = 5e^{-2t}$ $\quad r=-2$

$$p(s) = s^2 + 3s + 2 = (s+2)(s+1) = 0$$

$$s=-2 \quad s=-1$$

$$x_h(t) = c_1 e^{-2t} + c_2 e^{-t} \quad \text{Span}\{e^{-2t}, e^{-t}\}$$

ERF: $x_p(t) = \dfrac{5e^{-2t}}{p(-2)}$ $\quad p(-2) = 0$

ERF fails! $\longrightarrow$ RRF

LINEAR OPERATORS $\left[D^2 + 3D + 2I\right] x(t) = 5e^{-2t}$

$$(D+2I)\circ(D+I)\, x(t) = 5e^{-2t}$$

Note: $(D+2I)(e^{-2t}) = -2e^{-2t} + 2e^{-2t} = 0$

$$(D+2I)\left[D^2+3D+2I\right] x(t) = 5\cdot(D+2I)e^{-2t} = 0$$

$$\left[(D+2I)^2 \circ (D+I)\right] x(t) = 0 \quad \text{homog.}$$

char polynomial $(s+2)^2(s+1) = 0$

$$s=-2\ (Am=2), \quad s=-1\ (Am=1)$$

$$\downarrow \qquad\qquad \downarrow$$

$$e^{-2t}, te^{-2t} \qquad\qquad e^{-t}$$

part. solution

$$\Rightarrow \quad x(t) = \underline{c_1 e^{-2t}} + \boxed{c_2 te^{-2t}} + \underline{c_3 e^{-t}}$$

homog. solns

$x_p = \underline{c_2}\, te^{-2t}$

What is c_2?

Better approach: Resonant Response Formula (RRF)

SEE Notes #5, pages 5-6 for details

RRF $x_p = \dfrac{5te^{-2t}}{p'(-2)} = \dfrac{5te^{-2t}}{-1}$

$p(s) = s^2 + 3s + 2$ $\quad = \underline{\underline{-5te^{-2t}}}$

$p'(s) = 2s + 3 \quad p'(-2) = -1$

<u>TRUE RESONANCE:</u>

FRICTIONLESS SPRING. — HARMONIC MOTION.
FORCE IT w/ same frequency that spring naturally has.

<u>Ex:</u> $\ddot{x} + 9x = 0$ homog. $\left(\begin{array}{c}\text{natural} \\ \text{frequency} \\ = 3\end{array}\right)$

$p(s) = s^2 + 9 = 0 \rightarrow s = \pm 3i.$

$\{e^{3it}, e^{-3it}\} \rightarrow \{\cos 3t, \sin 3t\}$

$x_h(t) = C_1 \cos 3t + C_2 \sin 3t.$

MATCH natural frequency w/ oscill input

<u>ex:</u> $\boxed{\ddot{x} + 9x = 2\cos 3t}$

$x_h(t) = C_1 \cos 3t + C_2 \sin 3t$

<u>$x_p(t)$</u>

Complex Replacement

$\ddot{z} + 9z = 2e^{3it}$, Real part

ERF $z_p = \dfrac{2e^{3it}}{p(3i)}$ $\quad p(3i) = 0$ ERF FAILS,

RRF $z_p = \dfrac{2te^{3it}}{p'(3i)}$ $\quad p'(s) = 2s$, $p'(3i) = 6i$

$$z_p(t) = \frac{2t\,(\cos 3t + i \sin 3t)}{6i}\,\frac{i}{i}$$

$$= \frac{2t\,(-\sin 3t + i \cos 3t)}{-6}$$

Real part $\Rightarrow$ $\boxed{\frac{1}{3} t \sin 3t = x_p(t)}$

Also: ① Linear Time Invariant (LTI) ODES and example

② Handling "Problematic Inputs"

Ex: $y'' + 9y = xe^x \cos x$

ⓐ method of Undetermined Coefficients

ⓑ Variation of Parameters (2nd order VERSION)