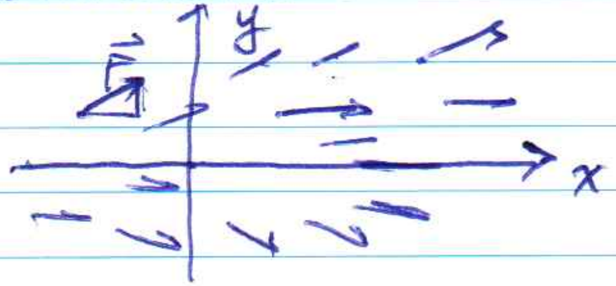
Lecture #10 - mostly from Lecture #10 NOTES [CONVOLUTION, ETC.]
ZSR + ZIR

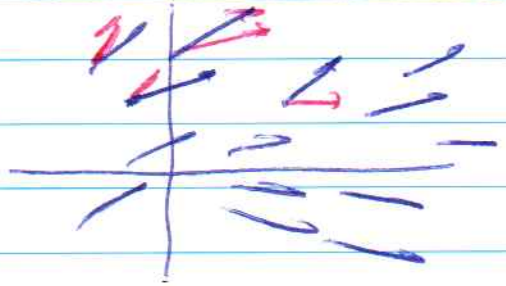
## vector fields, continuous dynamical systems and systems of 1st order ODE's.

Autonomous (time-Invariant) vector field.
(STATIC)




$$(x,y) \to \vec{F}(x,y) = \langle F_1(x,y), F_2(x,y) \rangle$$

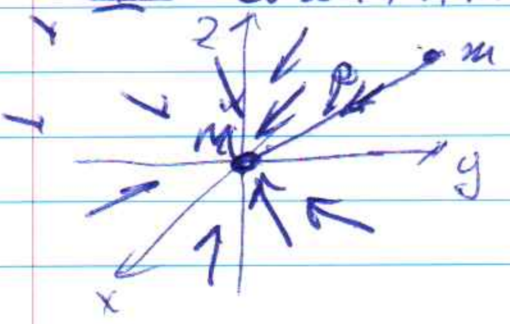
Nonautonomous vector field.



$$(x,y,t) \to \vec{F}(x,y,t)$$
$$= \langle F_1(x,y,t), F_2(x,y,t) \rangle$$

vector field changes in time

Ex: Gravitation attraction



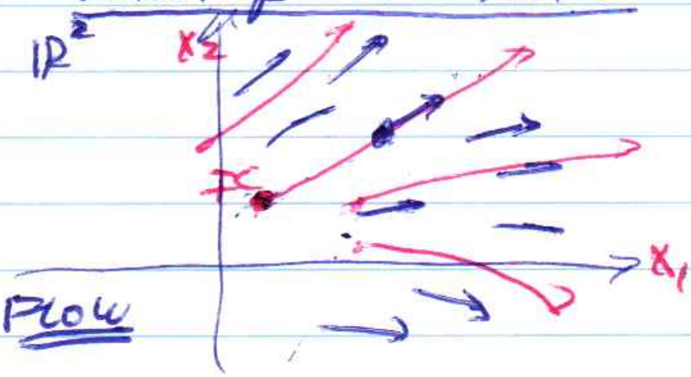

$$\vec{F} = -\left(\frac{GMm}{\rho^2}\right) \vec{u}_\rho$$

unit RADIAL VECTOR

$$= -\frac{GMm}{(x^2+y^2+z^2)} \frac{\langle x,y,z \rangle}{\sqrt{x^2+y^2+z^2}}$$

in Cartesian coordinates
(not advisable)

velocity vector field




$(x,y) \rightarrow \vec{v}(x,y)$

Find Trajectories (Solutions)

$\vec{x}(t) = \langle x_1(t), x_2(t) \rangle$

parameterized curve

Flow

$\vec{x}(0) = \langle x_1(0), x_2(0) \rangle$

Find $\vec{x}(t)$ such that at time $t$,

$$\boxed{\frac{d\vec{x}}{dt} = \vec{v}(\vec{x}(t))}$$

$$\vec{v}(\vec{x}) = \langle F_1(x_1, x_2), F_2(x_1, x_2) \rangle$$

$$\left\{ \begin{aligned} \frac{dx_1}{dt} &= F_1(x_1, x_2) \\ \frac{dx_2}{dt} &= F_2(x_1, x_2) \end{aligned} \right\}$$

system of 1st order ORD DIFF. EQNS

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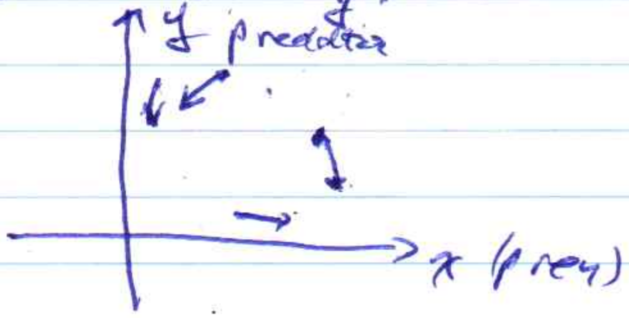
$\mathbb{R}^n$ $\quad \vec{v}(x_1, \ldots, x_n)$ $\quad$ Initial conditions $\langle x_1(0), x_2(0), x_3(0), \ldots, x_n(0) \rangle$

$$\Longrightarrow \vec{x}(t) = \langle x_1(t), x_2(t), \ldots, x_n(t) \rangle$$

$$\left\{ \begin{aligned} \frac{dx_1}{dt} &= F_1(x_1, \ldots, x_n) \\ &\vdots \\ \frac{dx_n}{dt} &= F_n(x_1, \ldots, x_n) \end{aligned} \right\}$$

Autonomous system of 1st order ODE's.

Predator Prey.




Reference : $\boxed{\text{PPLANE}}$

java-based program to show 1st order systems and trajectories [2 VARIABLES]

ODE and Reduction of Order.

nth order ODE $\longrightarrow$ system of n 1st order ODE's.

Ex: $\boxed{\ddot{x} + 3\dot{x} + 2x = 0}$ 2nd order homog.

$$\frac{dx}{dt} = \dot{x} = y \qquad \ddot{x} = \dot{y} = \frac{dy}{dt}$$

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -2x - 3y \end{cases}$$

Ex: $\ddot{x} + 3t^2\dot{x} + 2e^t x = 0$ 2nd. order

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -2e^t x - 3t^2 y \end{cases}$$

nonautonomous system of 1st order ODE's

Ex: $\dddot{x} + 3\ddot{x} - 4\dot{x} + 2x = 0$ $\quad \dot{x} = y \quad \ddot{x} = \dot{y} = z$

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = z \\ \dfrac{dz}{dt} = -2x + 4y - 3z \end{cases}$$

autonomous system of 1st od ODE's

## SPECIAL CASE

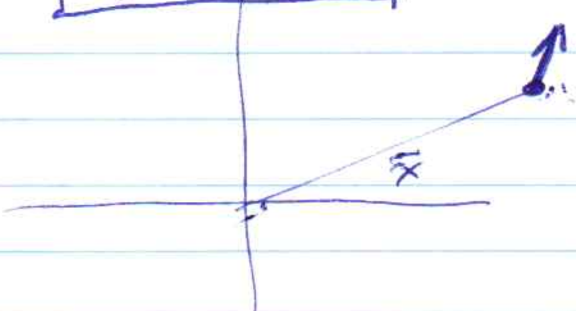
$$\left\{\begin{aligned}
\frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\
\frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\
&\vdots \\
\frac{dx_n}{dt} &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n
\end{aligned}\right\}$$

Linear functions

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \qquad \frac{d\vec{x}}{dt} = \begin{bmatrix} dx_1/dt \\ \vdots \\ dx_n/dt \end{bmatrix}$$

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} dx_1/dt \\ \vdots \\ dx_n/dt \end{bmatrix} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}}_{\vec{x}} = A\vec{x}$$

$$\boxed{\frac{d\vec{x}}{dt} = A\vec{x}}$$




we can always solve these explicitly.

SIMPLEST CASE $1 \times 1$

$$\frac{dx}{dt} = kx \qquad x(0) = x_0$$

$$\int \frac{dx}{x} = \int k\,dt \longrightarrow \ln|x| = kt + C$$

$$e \qquad e$$

$$x(t) = a e^{kt} \qquad x(0) = a$$

$$= x(0) e^{kt} = e^{kt} x(0)$$

uncoupled system $\quad x(0) = x_0,\ y(0) = y_0 \quad \vec{x}(0) = (x_0, y_0)$

$$\left\{ \begin{aligned} \frac{dx}{dt} &= k_1 x \\ \frac{dy}{dt} &= k_2 y \end{aligned} \right\} \longrightarrow \begin{aligned} x(t) &= e^{k_1 t} x(0) \\ y(t) &= e^{k_2 t} y(0) \end{aligned}$$

$$\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = D\vec{x} \qquad D \text{ DIAGONAL MATRIX}$$

$$\vec{x}(t) = \begin{bmatrix} e^{k_1 t} x(0) \\ e^{k_2 t} y(0) \end{bmatrix} = \begin{bmatrix} e^{k_1 t} & 0 \\ 0 & e^{k_2 t} \end{bmatrix} \begin{bmatrix} x(0) \\ y(0) \end{bmatrix}$$

$$\vec{x}(t) = \left[ e^{tD} \right] \vec{x}(0)$$

evolution MATRIX.

Promise:

Given $\frac{d\vec{x}}{dt} = A\vec{x}$, $\vec{x}(0)$

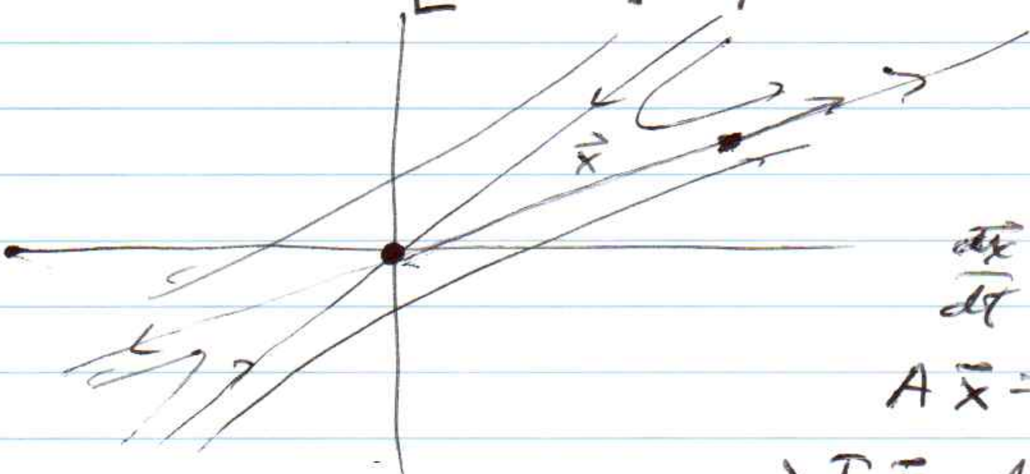
$$\Longrightarrow \quad x(t) = \left[ e^{tA} \right] \vec{x}(0)$$

Ex: $\begin{cases} \dfrac{dx}{dt} = 5x - 6y \\ \dfrac{dy}{dt} = 3x - 4y \end{cases}$ $x(0) = 3$, $y(0) = 1$

$\vec{x}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

Coupled system

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = A\vec{x}$$




SEEK INVARIANT DIRECTIONS

$\vec{v} \parallel \vec{x}$

$\frac{d\vec{x}}{dt} = Ax \parallel \vec{x}$

$A\vec{x} = \lambda \vec{x} = \lambda I \vec{x}$

$\lambda I \vec{x} - A\vec{x} = \vec{0}$

$$\boxed{(\lambda I - A)\vec{x} = \vec{0}}$$

$$\underbrace{\det(\lambda I - A)}_{\text{char. polyn. } P_A(\lambda)} = 0$$

SOLUTIONS ONLY WHEN $\lambda I - A$ is <u>not</u> invertible

CRITERION: $\lambda I - A$ invertible if and only if $\det(\lambda I - A) \neq 0$

ROOTS OF $P_A(\lambda) \longrightarrow$ eigenVALUES

If $\lambda_i$ and eigenVALUE

$\text{Ker}[\lambda_i I - A] \longrightarrow$ eigenvectors [INVARIANT DIRECTIONS]