Reminder: Changing coordinates

$$\begin{tikzcd} \vec{x} \in \{\mathbb{R}^n, \mathcal{E}\} \arrow[r, "A"] & \{\mathbb{R}^n, \mathcal{E}\} \\ \left[\vec{x}\right]_{\mathcal{B}} \in \{\mathbb{R}^n, \mathcal{B}\} \arrow[u, "S"] \arrow[r, "\left[A\right]_{\mathcal{B}}"] & \{\mathbb{R}^n, \mathcal{B}\} \arrow[u, "S"] \end{tikzcd}$$

$\mathcal{E} = \{\vec{e}_1, \cdots, \vec{e}_n\}$ STANDARD BASIS

$\mathcal{B} = \{\vec{v}_1, \cdots, \vec{v}_n\}$ alternate basis

If $\vec{x}$ (standard coords) is expressed as $\vec{x} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$ in other basis, so $\{c_1, \cdots, c_n\}$ are coords Relative to basis $\mathcal{B}$.

$$\vec{x} = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

$$\vec{x} = S\left[\vec{x}\right]_{\mathcal{B}}$$

FROM DIAGRAM ABOVE

$$\left[A\right]_{\mathcal{B}} = S^{-1}AS = B$$

$$\text{or } A = SBS^{-1}$$

We generally seek a basis $\mathcal{B}$ consisting of eigenvectors and other linearly independent vectors so that the MATRIX B is simple (standardized), e.g. in DIAGONAL FORM in the case of Real, distinct eigenvalues.

FOR A system of ODE's of form $\frac{d\vec{x}}{dt} = A\vec{x}$, $\vec{x}(0)$,

in new COORDINATES $\frac{d\vec{x}}{dt} = SBS^{-1}\vec{x}$ $\qquad \vec{u} = S^{-1}\vec{x} = \left[\vec{x}\right]_{\mathcal{B}}$

$$\Rightarrow S^{-1}\frac{d\vec{x}}{dt} = \frac{d}{dt}\left(S^{-1}\vec{x}\right) = B\left(S^{-1}\vec{x}\right) \Rightarrow \frac{d\vec{u}}{dt} = B\vec{u}$$

If $\left[e^{tB}\right]$ can be easily determined, then $\vec{u}(t) = \left[e^{tB}\right]\vec{u}(0)$

$$\Rightarrow S^{-1}\vec{x} = \left[e^{tB}\right]S^{-1}\vec{x}(0) \Rightarrow \vec{x}(t) = \underbrace{S\left[e^{tB}\right]S^{-1}}_{\left[e^{tA}\right]}\vec{x}(0)$$

$$\text{So } \vec{x}(t) = \left[e^{tA}\right]\vec{x}(0)$$

solves system

$$\begin{cases} \dfrac{dx}{dt} = 5x - 6y \\ \dfrac{dy}{dt} = 3x - 4y \end{cases} \qquad x(0) = 3, \; y(0) = 1$$

$$\frac{d\vec{x}}{dt} = A\vec{x} \qquad A = \begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix}$$

$$\lambda I - A = \begin{bmatrix} \lambda - 5 & 6 \\ -3 & \lambda + 4 \end{bmatrix} \qquad \vec{x}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$P_A(\lambda) = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1) = 0$$

$\boxed{\text{See Lecture Notes \#12}}$

$$\lambda_1 = 2 \qquad \lambda_1 = -1$$

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$S = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \qquad S^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

$$S^{-1}AS = D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\frac{d\vec{x}}{dt} = A\vec{x} \qquad A = SDS^{-1}$$

$$\frac{d\vec{x}}{dt} = SDS^{-1}\vec{x} \qquad \frac{d}{dt}(\overset{\vec{u}}{S^{-1}\vec{x}}) = D(\overset{\vec{u}}{S^{-1}\vec{x}})$$

$$\boxed{\frac{d\vec{u}}{dt} = D\vec{u}} \rightarrow u(t) = \left[e^{tD}\right]\vec{u}(0)$$

$$\vec{u} = S^{-1}x \qquad S^{-1}\vec{x} = \left[e^{tD}\right]S^{-1}\vec{x}(0)$$

$$\vec{x} = S\vec{u} \qquad \vec{x}(t) = \underbrace{S\left[e^{tD}\right]S^{-1}}_{\left[e^{tA}\right]}\vec{x}(0)$$

<u>evolution MATRIX</u>

$$\frac{d\vec{x}}{dt} = A\vec{x} \qquad S^{-1}AS = D$$

$$A = SDS^{-1}$$

$$\left[e^{tA}\right] = S\left[e^{tD}\right]S^{-1}$$

$$\vec{x}(t) = \left[e^{tA}\right]\vec{x}(0) = S\left[e^{tD}\right]S^{-1}\vec{x}(0)$$

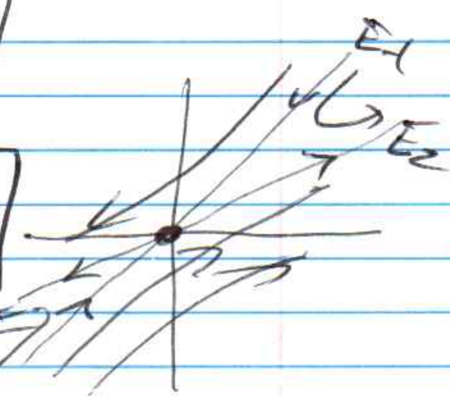
$$\vec{x}(t) = \begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}e^{2t} & 0\\ 0 & e^{-t}\end{bmatrix}\underbrace{\begin{bmatrix}1 & -1\\ -1 & 2\end{bmatrix}\begin{bmatrix}3\\ 1\end{bmatrix}}_{\begin{bmatrix}c_1\\ c_2\end{bmatrix}}$$

$$= \begin{bmatrix}2e^{2t} & e^{-t}\\ e^{2t} & e^{-t}\end{bmatrix}\begin{bmatrix}2\\ -1\end{bmatrix}$$

$\boxed{\text{See PPLANE}}$

$$\vec{x}(t) = \begin{bmatrix}4e^{2t} - e^{-t}\\ 2e^{2t} - e^{-t}\end{bmatrix} = \begin{bmatrix}x(t)\\ y(t)\end{bmatrix}$$

$$= 2e^{2t}\underbrace{\begin{bmatrix}2\\ 1\end{bmatrix}}_{\vec{v}_1} - e^{-t}\underbrace{\begin{bmatrix}1\\ 1\end{bmatrix}}_{\vec{v}_2}$$



$$\ddot{x} + \omega^2 x = 0$$

HARMONIC OSCILLATOR

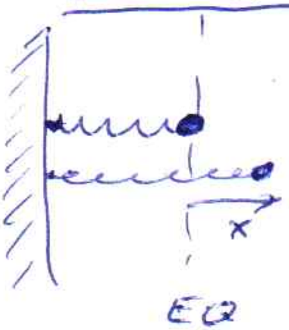
$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = -\omega^2 x$$

$$\left\{\begin{aligned}\frac{dx}{dt} &= y\\ \frac{dy}{dt} &= -\omega^2 x\end{aligned}\right\}$$

$$\vec{x} = \begin{bmatrix}x\\ y\end{bmatrix} \qquad \frac{d\vec{x}}{dt} = A\vec{x} \qquad A = \begin{bmatrix}0 & 1\\ -\omega^2 & 0\end{bmatrix}$$

Harmonic Oscillator



$\dot{x} = v$

$\ddot{x} = a$

$ma = F = -kx - cv$

$ma + cv + kx = 0$

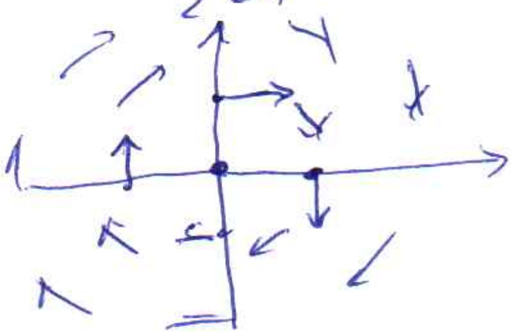
$m\ddot{x} + c\dot{x} + kx = 0$

$$\boxed{\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0}$$

2nd order ODE

$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -\omega^2 x \end{cases}$$

$\omega = 1$
$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -x \end{cases}$$



Reduction of order

$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -\frac{k}{m}x - \frac{c}{m}y \end{cases}$$

Special case:
FRICTIONLESS
$c = 0$

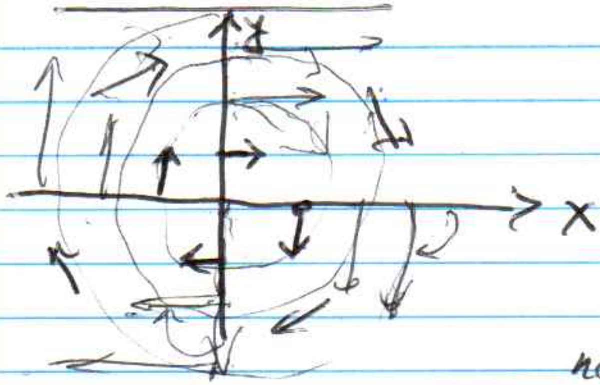
$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -\frac{k}{m}x \end{cases}$$

$\frac{k}{m} = \omega^2$

$$\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{d\vec{x}}{dt}$$

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Vector field:



NO INVARIANT DIRECTIONS

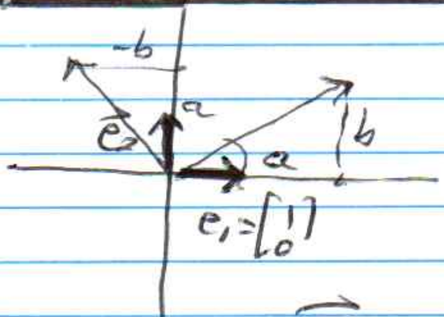
No Real eigenvectors

no Real eigenvalues

Algebraic Formalism.

$S^{-1}AS = B$ still works.

$\boxed{\text{ASIDE : ROTATION-DILATION MATRIX.}}$



$$\vec{e}_1 \to \begin{bmatrix} a \\ b \end{bmatrix} \qquad \vec{e}_2 \to \begin{bmatrix} -b \\ a \end{bmatrix} \quad \to B = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

GOAL: $\dfrac{d\vec{x}}{dt} = A\vec{x}$

$$S'^{-1}AS' = B$$

$$A = SBS^{-1}$$

$$\left[e^{tA}\right] = S\left[e^{tB}\right]S^{-1}$$

$$\vec{x}(t) = \left[e^{tA}\right]\vec{x}(0)$$

$$= S\left[e^{tB}\right]S^{-1}\,\vec{x}(0)$$

$$A=\begin{bmatrix}0 & 1\\ -\omega^2 & 0\end{bmatrix} \qquad \lambda I - A = \begin{bmatrix}\lambda & -1\\ \omega^2 & \lambda\end{bmatrix}$$

$$p_A(\lambda) = \lambda^2 + \omega^2 = 0$$

$$\lambda^2 = -\omega^2$$

$$\boxed{\begin{aligned}\lambda &= i\omega\\ \bar{\lambda} &= -i\omega\end{aligned}}$$

Formally

$\lambda = i\omega$

$$\begin{bmatrix}i\omega & -1\\ \omega^2 & i\omega\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$$

$$i\omega\alpha - \beta = 0 \longrightarrow \beta = i\omega\alpha$$

$$\alpha\omega^2 + i\omega\beta = 0$$

$$\alpha = 1 \qquad \beta = i\omega$$

complex eigenvector $w = \begin{bmatrix}1\\ i\omega\end{bmatrix}$

FACT: If $\vec{w}$ is a complex e-vector assoc with cx e-value $\lambda = a+ib$

then $\hat{\vec{w}}$ is a complex e-vector assoc with cx e-value $\bar{\lambda} = a-ib$.

REASON

$$\Downarrow$$

$$\begin{bmatrix}A\vec{w} = \lambda\vec{w}\\ \overline{A\vec{w}} = \overline{\lambda\vec{w}}\\ \Downarrow\\ A\hat{\vec{w}} = \bar{\lambda}\hat{\vec{w}}\end{bmatrix}$$

$$\lambda = a+ib \longrightarrow \vec{w} = \begin{bmatrix}\alpha\\ \beta\end{bmatrix}$$

$$\bar{\lambda} = a-ib \longrightarrow \hat{\vec{w}} = \begin{bmatrix}\bar{\alpha}\\ \bar{\beta}\end{bmatrix}$$

$$\lambda = i\omega \rightarrow \vec{w} = \begin{bmatrix}1\\ i\omega\end{bmatrix} = \begin{bmatrix}1\\ 0\end{bmatrix} + i\begin{bmatrix}0\\ \omega\end{bmatrix} = \vec{u} + i\vec{v}$$

$$\bar{\lambda} = -i\omega \rightarrow \hat{\vec{w}} = \begin{bmatrix}1\\ -i\omega\end{bmatrix} = \begin{bmatrix}1\\ 0\end{bmatrix} - i\begin{bmatrix}0\\ \omega\end{bmatrix} = \vec{u} - i\vec{v}$$

Formally $\{\lambda, \bar{\lambda}\} \to \{\vec{w}, \hat{\vec{w}}\}$

$$S = [\vec{w} \ \hat{\vec{w}}], \quad S^{-1}$$

$$S^{-1}AS = D = \begin{bmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{bmatrix}$$

could proceed as in Real e-value case,

If $\boxed{\lambda = a + ib} \to \boxed{\vec{w} = \vec{u} + i\vec{v}}$

$$A\vec{w} = \lambda\vec{w}$$
$$A\hat{\vec{w}} = \bar{\lambda}\hat{\vec{w}}$$

$$A(\vec{u} + i\vec{v}) = (a+ib)(\vec{u} + i\vec{v}) = (a\vec{u} - b\vec{v}) + i(b\vec{u} + a\vec{v})$$

$$A(\vec{u} - i\vec{v}) = (a-ib)(\vec{u} - i\vec{v}) = (a\vec{u} - b\vec{v}) - i(b\vec{u} + a\vec{v})$$

$$A\vec{u} + iA\vec{v} = (a\vec{u} - b\vec{v}) + i(b\vec{u} + a\vec{v})$$
$$A\vec{u} - iA\vec{v} = (a\vec{u} - b\vec{v}) - i(b\vec{u} + a\vec{v})$$

Add: $\cancel{2}A\vec{u} = \cancel{2}(a\vec{u} - b\vec{v})$ $\quad \boxed{A\vec{u} = a\vec{u} - b\vec{v}}$

Subtract: $\cancel{2i}A\vec{v} = \cancel{2i}(b\vec{u} + a\vec{v})$ $\quad \boxed{A\vec{v} = b\vec{u} + a\vec{v}}$

$$\text{Span}\{\vec{u}, \vec{v}\} \xrightarrow{A} \text{Span}\{\vec{u}, \vec{v}\}$$

INVARIANT SUBSPACE $\{\vec{v}, \vec{u}\}$

$$\begin{Bmatrix} A\vec{v} = a\vec{v} + b\vec{u} \\ A\vec{u} = -b\vec{v} + a\vec{u} \end{Bmatrix}$$

$\{\vec{v}, \vec{u}\}$ BASIS for new COORDS

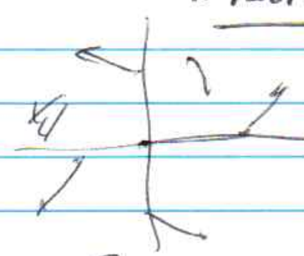
$$S = \begin{bmatrix} \vec{v} & \vec{u} \end{bmatrix} \qquad S\vec{e}_1 = \vec{v}$$
$$S\vec{e}_2 = \vec{u}$$

$$\left\{ \begin{aligned} AS\vec{e}_1 &= aS\vec{e}_1 + bS\vec{e}_2 \\ AS\vec{e}_2 &= -bS\vec{e}_1 + aS\vec{e}_2 \end{aligned} \right\}$$

$$(S^{-1}AS)\underline{\vec{e}_1} = a\vec{e}_1 + b\vec{e}_2 = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$(S^{-1}AS)\underline{\vec{e}_2} = -b\vec{e}_1 + a\vec{e}_2 = \begin{bmatrix} -b \\ a \end{bmatrix}$$

$$S^{-1}AS = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = B \qquad \text{Rotation} \ \underline{\text{Dilation}}$$



$$A = SBS^{-1}$$

$$\left[e^{tA}\right] = S\left[e^{tB}\right]S^{-1}$$

$$x(t) = \left[e^{tA}\right]\vec{x}(0) = S\left[e^{tB}\right]S^{-1}\vec{x}(0)$$

$$\boxed{\left[e^{tB}\right] = \ ?} \qquad \lambda = a + ib$$

<u>Refer to lecture notes</u> for Calculations.

$$\text{Result: } \left[e^{tB}\right] = e^{at}\begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix}$$

exponential growth or decay

← time-varying ROTATION MATRIX

[a=0 → periodic orbit]

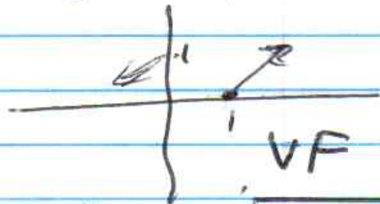
Together ⟹ spiral trajectories

Examples

$$\begin{cases} \dfrac{dx}{dt} = 2x - 5y \\ \dfrac{dy}{dt} = 2x - 4y \end{cases} \qquad x(0) = 0 \quad y(0) = 1$$

$$\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix} \quad \frac{d\vec{x}}{dt} = A\vec{x} \quad A = \begin{bmatrix} 2 & -5 \\ 2 & -4 \end{bmatrix}$$

$$\lambda I - A = \begin{bmatrix} \lambda-2 & 5 \\ -2 & \lambda+4 \end{bmatrix}$$



$$P_A(\lambda) = \lambda^2 + 2\lambda + 2 = 0$$

$$\lambda = \frac{-2 \pm \sqrt{4-8}}{2}$$

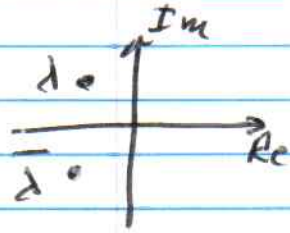
$$= -1 \pm i$$

Use vector field to distinguish clockwise vs. counterclockwise

$$(\lambda+1)^2 + 1 = 0$$

$$\lambda = -1 \pm i$$

$$\lambda = -1 + i$$
$$\bar{\lambda} = -1 - i$$



$$\lambda = -1 + i$$

Cx e-values

$$\lambda = -1 + i = a + ib.$$
$$a = -1 < 0$$
decay

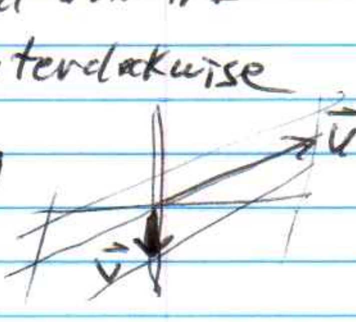
Spiral INWARD counterclockwise

$$\begin{bmatrix} -3+i & 5 \\ -2 & 3+i \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(-3+i)\alpha + 5\beta = 0$$

$$5\beta = (3-i)\alpha$$

$$\vec{w} = \begin{bmatrix} 5 \\ 3-i \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix} + i \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$
$$= \vec{u} + i\vec{v}$$



$\{\vec{v}, \vec{u}\} = \mathcal{B}$ $\qquad \lambda = \underline{\underline{-1}} + \underline{\underline{i}}$ $\quad a=-1$, $b=1$

$$S = \begin{bmatrix} 0 & 5 \\ -1 & 3 \end{bmatrix} \qquad S^{-1} = \frac{1}{5}\begin{bmatrix} 3 & -5 \\ 1 & 0 \end{bmatrix}$$

$$S^{-1}AS = B = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$$

$$A = SBS^{-1}$$

$$\left[e^{tA}\right] = S\left[e^{tB}\right]S^{-1}$$

$$\checkmark \qquad \checkmark$$

$$\left[e^{tB}\right] = e^{-t}\begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$$

$$\vec{x}(t) = \left[e^{tA}\right]\vec{x}(0) = \begin{bmatrix} 0 & 5 \\ -1 & 3 \end{bmatrix} \frac{e^{-t}}{5}\begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}\begin{bmatrix} 3 & -5 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{e^{-t}}{5}\begin{bmatrix} 5\sin t & 5\cos t \\ -\cos t + 3\sin t & \sin t + 3\cos t \end{bmatrix}\begin{bmatrix} -5 \\ 0 \end{bmatrix}$$

$$= -e^{-t}\begin{bmatrix} 5\sin t & 5\cos t \\ -\cos t + 3\sin t & \sin t + 3\cos t \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= -e^{-t}\begin{bmatrix} 5\sin t \\ -\cos t + 3\sin t \end{bmatrix} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -4x+4y \end{cases}$$

Repeated (real) eigenvalue case

$$\frac{d\vec{x}}{dt} = A\vec{x} \qquad A = \begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix}$$

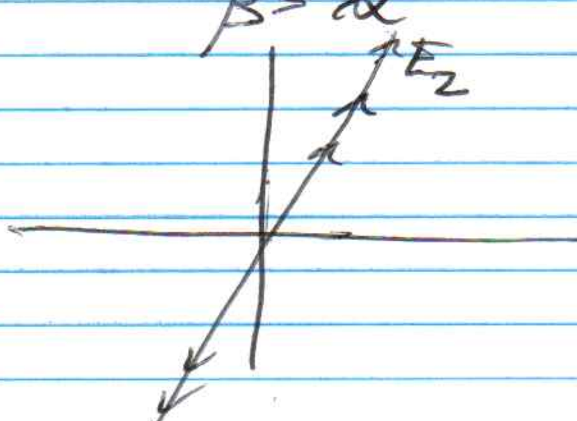
$$\lambda I - A = \begin{bmatrix} \lambda & -1 \\ 4 & \lambda-4 \end{bmatrix}$$

$$P(\lambda) = \lambda^2 - 4\lambda + 4 = 0$$
$$= (\lambda-2)^2 = 0 \qquad \lambda_1 = 2$$

AM = 2
GM = 1

$$\lambda_1 = 2 \rightarrow \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{matrix} 0 \\ 0 \end{matrix}$$

$$2\alpha - \beta = 0 \quad \Rightarrow \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$
$$\beta = 2\alpha$$



only one eigenvector

generalized e-vector will be needed (next lecture)

$A\vec{v}_1 = \lambda \vec{v}_1 = \lambda I \vec{v}_1$

$(A - \lambda I)\vec{v}_1 = \vec{0}$ $\quad \vec{v}_1 \in \text{Ker}(A - \lambda I)$.

CAN'T FIND $\vec{v}_2$

Seek generalized eigenvector $\vec{v}_2$

$$* \begin{Bmatrix} A\vec{v}_1 = \lambda \vec{v}_1 \\ \underline{A\vec{v}_2 = \vec{v}_1 + \lambda \vec{v}_2} \end{Bmatrix} \quad \text{why?}$$

$\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$ $\quad [A]_{\mathcal{B}} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$

$S = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}$ $\quad S^{-1}$

$$\longrightarrow [A]_{\mathcal{B}} = S^{-1}AS = B = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

* <u>Note:</u> $A\vec{v}_2 - \lambda I \vec{v}_2 = (A - \lambda I)\vec{v}_2 = \vec{v}_1$

$(A - \lambda I)^2 \vec{v}_2 = (A - \lambda I)\vec{v}_1 = \vec{0}$

$\vec{v} \in \text{Ker}(A - \lambda I) \longrightarrow$ e-vector

$\vec{v} \in \text{Ker}(A - \lambda I)^k \longrightarrow$ generalized eigenvector

$A = SBS^{-1}$

$[e^{tA}] = S[e^{tB}]S^{-1}$ $\quad [e^{tB}] = ?$

$(A - \lambda I)\vec{v}_2 = \vec{v}_1$.

$$\boxed{(\lambda I - A)\vec{v}_2 = -\vec{v}_1}$$

$$\begin{cases} \dfrac{dx}{dt} = 2x + y \\ \dfrac{dy}{dt} = \quad 2y + z \\ \dfrac{dz}{dt} = \quad\quad 2z \end{cases}$$

① $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} = 2I + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

② vs. $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & a & 2 \end{bmatrix}$

③ vs. $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = D$

① $\lambda I - A = \begin{bmatrix} \lambda-2 & -1 & 0 \\ 0 & \lambda-2 & -1 \\ 0 & 0 & \lambda-2 \end{bmatrix}$ $\quad P_A(\lambda) = (\lambda-2)^3$

$\lambda = 2$ $AM = 3$

$GM = 1$

② $\lambda I - A = \begin{bmatrix} \lambda-2 & -1 & 0 \\ 0 & \lambda-2 & 0 \\ 0 & 0 & \lambda-2 \end{bmatrix}$ $\quad P_A(\lambda) = (\lambda-2)^3 = 0$

$\lambda = 2$, $AM = 3$

$GM = 2$

③ $\lambda I - \sim\sim \rightarrow$ $P_A(\lambda) = (\lambda-2)^3$

$\lambda = 2$ $AM = 3$

$GM = 3$

DIAGONALIZABLE