COUPLED SYSTEMS – MATRIX and non-matrix solutions, evolution matrices

Solve

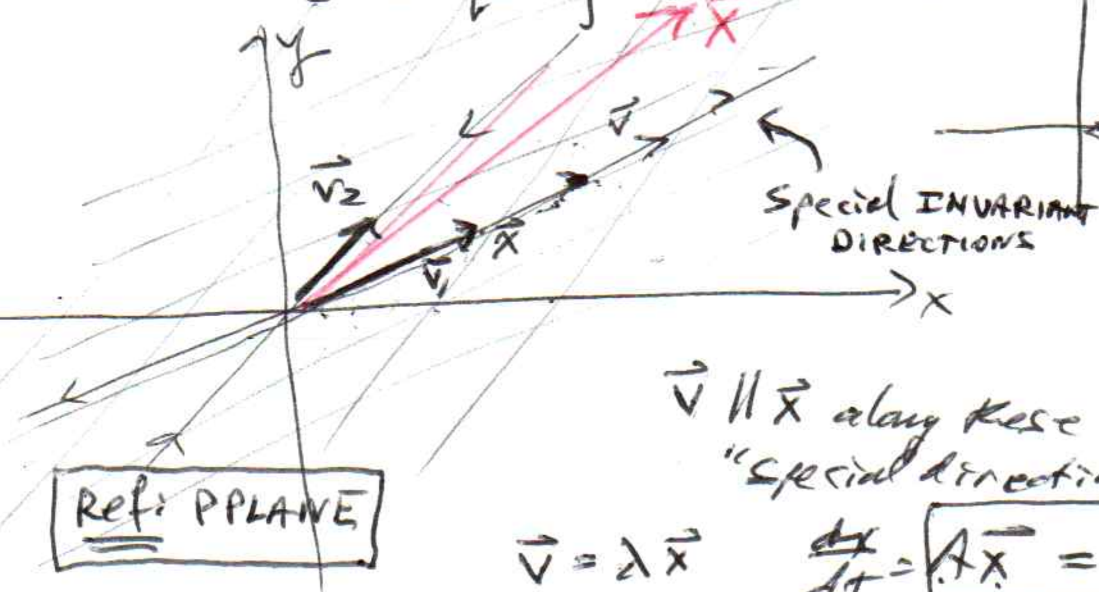
$$\left\{\begin{aligned} \frac{dx}{dt} &= 5x-6y \\ \frac{dy}{dt} &= 3x-4y \end{aligned}\right\} \qquad A = \begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

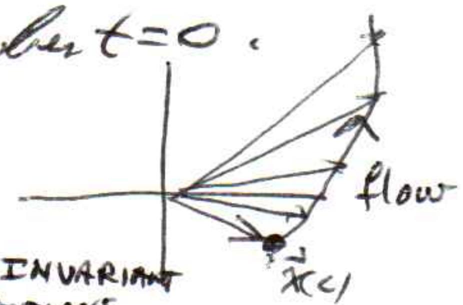
$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = A\vec{x} \qquad \frac{d\vec{x}}{dt} = A\vec{x}$$

Initial conditions $x(0)=3$, $y(0)=1$

Promise: Always solve this as $\vec{x}(t) = [e^{tA}]\,\vec{x}(0)$

$t=0$ $[e^{tA}] \longrightarrow I$ when $t=0$.





Ref: PPLANE

$\vec{v} \parallel \vec{x}$ along these "special direction"

$\vec{v} = \lambda\vec{x}$ $\quad \frac{d\vec{x}}{dt} = \boxed{A\vec{x} = \lambda\vec{x}}$

λ eigenvalue (characteristic value)

$A\vec{x} = \lambda I\vec{x}$

$A\vec{x} - \lambda I\vec{x} = 0$ $\qquad \lambda I\vec{x} - A\vec{x} = \vec{0}$ $\qquad \vec{x}$ eigenvector

$(A-\lambda I)\vec{x} = \vec{0}$ $\qquad \boxed{(\lambda I - A)\vec{x} = \vec{0}}$

only have solns if $\lambda I - A$ is not invertible

Linear Algebra: An $n\times n$ matrix A is invertible if and only if $\det A \neq 0$. ← $P(\lambda)$ characteristic polynomial

Seek solns when $\det(\lambda I - A) = 0$

$$A = \begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix} \qquad A - \lambda I = \begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 5-\lambda & -6 \\ 3 & -4-\lambda \end{bmatrix}$$

etc.

$\lambda I - A$

$$= \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda-5 & 6 \\ -3 & \lambda+4 \end{bmatrix}$$

$$\left| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right| = ad - bc$$

$$p(\lambda) = \lambda^2 - \lambda - 20 + 18$$
$$= \lambda^2 - \lambda - 2 = 0$$
$$= (\lambda - 2)(\lambda + 1) = 0$$

$\lambda_1 = 2$ growth $\qquad \lambda_2 = -1$ decay.

$\lambda_1 = 2 \quad \begin{bmatrix} -3 & 6 \\ -3 & 6 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$-3\alpha + 6\beta = 0 \rightarrow \alpha = 2\beta \qquad \beta = 1 \rightarrow \alpha = 2 \qquad \vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$

$\lambda_2 = -1 \quad \begin{bmatrix} -6 & 6 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$

$\alpha = \beta$

DISTINCT, REAL EIGENVALUES.

Solve system w/o MATRIX.

[COORDINATES RELATIVE TO A BASIS] $\vec{x} = c_1 \vec{v}_1 + c_2 \vec{v}_2 = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = S [\vec{x}]_{\mathcal{B}}$

[change of basis matrix]

$\mathcal{B} = \{\vec{v}_1, \vec{v}_2\} \qquad [\vec{x}]_{\mathcal{B}} = S^{-1} \vec{x}$

[coord vector rel. to $\mathcal{B}$]

Soln $\vec{x}(t) = c_1(t) \vec{v}_1 + c_2(t) \vec{v}_2$

$$\frac{d\vec{x}}{dt} = \frac{dc_1}{dt} \vec{v}_1 + \frac{dc_2}{dt} \vec{v}_2 = A\vec{x} = A(c_1 \vec{v}_1 + c_2 \vec{v}_2)$$
$$= c_1 A\vec{v}_1 + c_2 A\vec{v}_2$$
$$= c_1 \lambda_1 \vec{v}_1 + c_2 \lambda_2 \vec{v}_2.$$

So $\frac{dc_1}{dt} = \lambda_1 c_1, \quad \frac{dc_2}{dt} = \lambda_2 c_2 \quad \Rightarrow$

$$\Rightarrow c_1(t) = c_1(0)e^{\lambda_1 t} \qquad c_2(t) = c_2(0)\, e^{\lambda_2 t}$$

$$\vec{x}(t) = c_1(0)e^{\lambda_1 t}\,\vec{v}_1 + c_2(0)e^{\lambda_2 t}\,\vec{v}_2$$

Initial conditions $\vec{x}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$$\vec{x}(0) = c_1\vec{v}_1 + c_2\vec{v}_2 = c_1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$2c_1 + c_2 = 3$$
$$c_1 + c_2 = 1$$
$$c_1 = 2 \qquad c_2 = -1$$

$$\vec{x}(t) = 2e^{2t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} - e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

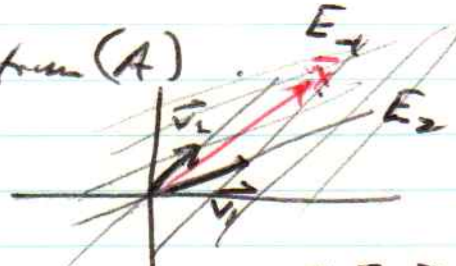
$$= \begin{bmatrix} 4e^{2t} - e^{-t} \\ 2e^{2t} - e^{-t} \end{bmatrix} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

MATRIX FORMULATION

Solve $\frac{d\vec{x}}{dt} = A\vec{x}$, $\vec{x}(0)$

Found $\{\lambda_1, \lambda_2\} = \text{Spectrum}(A)$

$\rightarrow \{\vec{v}_1, \vec{v}_2\} = \mathcal{B}$



$$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2$$

$$\vec{x} = c_1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2c_1 + c_2 \\ c_1 + c_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

standard

change-of-basis matrix S $\quad [\vec{x}]_{\mathcal{B}}$

$$\vec{x} = S[\vec{x}]_{\mathcal{B}} \qquad \vec{u} = [\vec{x}]_{\mathcal{B}} \qquad \vec{x} = S\vec{u}$$

$$S^{-1}\vec{x} = [\vec{x}]_{\mathcal{B}} \qquad\qquad S^{-1}\vec{x} = \vec{u}$$

$$\frac{d\vec{x}}{dt} = A\vec{x}$$

$$\frac{d}{dt}(S\vec{u}) = AS\vec{u} \quad *$$

$$S = [\vec{v}_1 \ \vec{v}_2]$$

S a matrix of constants

$$\left\{ \begin{array}{l} S\vec{e}_1 = \vec{v}_1 \\ S\vec{e}_2 = \vec{v}_2 \end{array} \right\}$$

<u>Lemma:</u> $\frac{d}{dt}[S\vec{u}] = S\frac{d\vec{u}}{dt}$

<u>Why?</u> $\vec{u} = \begin{bmatrix} u \\ v \end{bmatrix} \quad S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$S\vec{u} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} au+bv \\ cu+dv \end{bmatrix}$$

$$\frac{d}{dt}[S\vec{u}] = \frac{d}{dt}\begin{bmatrix} au+bv \\ cu+dv \end{bmatrix} = \begin{bmatrix} a\frac{du}{dt} + b\frac{dv}{dt} \\ c\frac{du}{dt} + d\frac{dv}{dt} \end{bmatrix}$$

$$= \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} du/dt \\ dv/dt \end{bmatrix}$$

$$* \quad S\frac{d\vec{u}}{dt} = AS\vec{u}$$

$$\boxed{\frac{d\vec{u}}{dt} = D\vec{u}}$$

$$\frac{d\vec{u}}{dt} = \underline{(S^{-1}AS)}\vec{u} = D\vec{u}$$

$$\vec{u}(t) = [e^{tD}]\vec{u}(0)$$

E-vectors $\{\vec{v}_1, \vec{v}_2\}$

$$\left\{ \begin{array}{l} A\vec{v}_1 = \lambda_1 \vec{v}_1 \\ A\vec{v}_2 = \lambda_2 \vec{v}_2 \end{array} \right\} \quad \left\{ \begin{array}{l} AS\vec{e}_1 = \lambda_1 S\vec{e}_1 \\ AS\vec{e}_2 = \lambda_2 S\vec{e}_2 \end{array} \right\}$$

$$\left\{ \begin{array}{l} S^{-1}AS\,\vec{e}_1 = \lambda_1 \vec{e}_1 \\ S^{-1}AS\,\vec{e}_2 = \lambda_2 \vec{e}_2 \end{array} \right\} \quad S^{-1}AS = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = D$$

<u>DIAGONAL</u>

$$\vec{u}(t) = \left[e^{tD}\right]\vec{u}(0)$$

$$S^{-1}\vec{x}(t) = \left[e^{tD}\right]S^{-1}\vec{x}(0)$$

$$\vec{x}(t) = \underbrace{S\left[e^{tD}\right]S^{-1}}_{\left[e^{tA}\right]}\vec{x}(0) = \left[e^{tA}\right]\vec{x}(0)$$

Define this evolution matrix by this.

QUICK RECAP

$$S^{-1}AS = D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

$$A = SDS^{-1}$$

$$\left[e^{tA}\right] = S\left[e^{tD}\right]S^{-1} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

In example

$$\vec{x}(t) = \left[e^{tA}\right]\vec{x}(0) = S\left[e^{tD}\right]S^{-1}\vec{x}(0) \leftarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{2t} & e^{-t} \\ e^{2t} & e^{-t} \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 4e^{2t} - e^{-t} \\ 2e^{2t} - e^{-t} \end{bmatrix}$$

$$= 2e^{2t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} - e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$$

Changing Coordinates (in General)

function $f(x) = x^2 \rightarrow y = x^2$

$$\begin{tikzcd}
\text{feet} \quad x \arrow[r, "f"] & y \quad \text{(pounds)} \\
\text{inches} \quad u \arrow[u, "g"] \arrow[r, "\tilde{f}"] & v \quad \text{ounces} \arrow[u, "h"']
\end{tikzcd}$$

x feet

$v = 16y$

$u = 12x$ inches $\quad x = \frac{u}{12}$

$x = g(u)$

$y = \frac{v}{16}$

$= h(v)$

$y = x^2$

$\frac{v}{16} = \left(\frac{u}{12}\right)^2 \qquad v = 16 \cdot \left(\frac{u}{12}\right)^2$

$\tilde{f}(u) = \left[h^{-1} f \circ g\right](u)$ $\quad$ [Representing a function in different coordinates (units)]

IN THE CASE OF A (SQUARE) MATRIX

$$\begin{tikzcd}
\vec{x} \in \{\mathbb{R}^n, \mathcal{E}\} \arrow[r, "A"] & \{\mathbb{R}^n, \mathcal{E}\} \quad A\vec{x} \\
{[\vec{x}]_{\mathcal{B}}} \in \{\mathbb{R}^n, \mathcal{B}\} \arrow[u, "S"'] \arrow[r, "\tilde{A}", "{[A]_{\mathcal{B}}}"'] & \{\mathbb{R}^n, \mathcal{B}\} \arrow[u, "S"]
\end{tikzcd}$$

$A\vec{v}_1 = \lambda_1 \vec{v}_1$

$A\vec{v}_2 = \lambda_2 \vec{v}_2$

$[A]_{\mathcal{B}} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

$= D$

$$\tilde{A} = \boxed{[A]_{\mathcal{B}} = S^{-1}AS}$$