Repeated eigenvalues

2nd order ODE: $\ddot{x} + 4\dot{x} + 4x = 0$ $\quad x(0) = 3$, $x'(0) = 2$

$x = e^{rt}$

characteristic polynomial: $\boxed{p(r) = r^2 + 4r + 4} = (r+2)^2$

$r = -2 \quad Am = 2$

$\Rightarrow \{e^{-2t}, te^{-2t}\}$

$x(t) = c_1 e^{-2t} + c_2 t e^{-2t}$, etc.

Reduction of order:
$$\left\{\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -4x - 4y \end{aligned}\right\} \text{phase space}$$

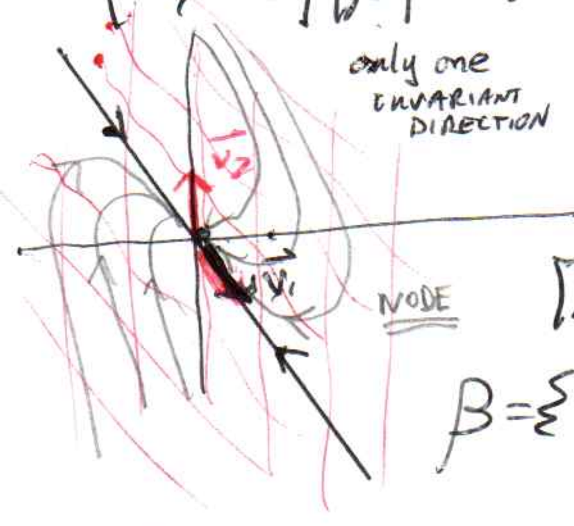
$\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$

$\frac{d\vec{x}}{dt} = A\vec{x} \qquad A = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix} \qquad \vec{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$\lambda I - A = \begin{bmatrix} \lambda & -1 \\ 4 & \lambda+4 \end{bmatrix} \qquad \boxed{p(\lambda) = \lambda^2 + 4\lambda + 4} = (\lambda+2)^2 = 0$

evals: $\lambda = -2 \quad Am = 2$

$\begin{bmatrix} -2 & -1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad -2\alpha - \beta = 0 \qquad \beta = -2\alpha \qquad \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \vec{v}_1$



only one INVARIANT DIRECTION

NODE

Generalized e-vector
$$\left\{\begin{aligned} A\vec{v}_1 &= \lambda \vec{v}_1 \\ A\vec{v}_2 &= \vec{v}_1 + \lambda \vec{v}_2 \end{aligned}\right\}$$

$[A]_{\mathcal{B}} \qquad S = [\vec{v}_1 \ \vec{v}_2]$

$[A]_\beta = S^{-1}AS = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} = B$

$\beta = \{\vec{v}_1, \vec{v}_2\}$

$$\frac{d\vec{x}}{dt} = A\vec{x} \longrightarrow \vec{x}(t) = [e^{tA}]\vec{x}(0)$$

$$S^{-1}AS = B \qquad [e^{tA}] = S[e^{tB}]S^{-1}$$

$$A = SBS^{-1} \qquad \vec{x}(t) = \underbrace{S[e^{tB}]}_{\underline{\underline{M(t)}}}\underbrace{S^{-1}\vec{x}(0)}_{[\vec{x}(0)]_{\mathcal{B}}}$$

Fundamental MATRIX.

$$A\vec{v}_2 = \vec{v}_1 + \lambda\vec{v}_2 \qquad \vec{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$(\lambda I - A)\vec{v}_2 = -\vec{v}_1$$

$$\begin{bmatrix} -2 & -1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \qquad \alpha = 1$$

$$-2\alpha - \beta = -1 \qquad \beta = -2\alpha + 1 = -1$$

$$4\alpha + 2\beta = 2 \qquad \cancel{\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}}$$

We need a generalized eigenvector, but there are many good choices.

$$\boxed{\text{OR}} \quad \alpha = 0 \rightarrow \beta = 1 \qquad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$S = [\vec{v}_1 \ \vec{v}_2] = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \qquad S^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$S^{-1}AS = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix} = B$$

<u>Q</u>: If $B = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$, what is $[e^{tB}]$?

<u>A</u>: $[e^{tB}] = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix} = e^{\lambda t}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$

<u>Why?</u> <u>HW exercise</u> $B = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \lambda I = A + \lambda I$$

Solve $\frac{d\vec{x}}{dt} = A\vec{x}$ with $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\boxed{\text{What is } \left[e^{tA}\right]?}$

$$\left\{\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= 0 \end{aligned}\right\} \begin{aligned} &= c_1 \rightarrow x(t) = c_1 t + c_2 \\ &\qquad\quad x(0) = c_2 \\ &\longrightarrow y(t) = c_1 = y(0) \end{aligned}$$

$$\Rightarrow \left\{\begin{aligned} x(t) &= x(0) + t\,y(0) \\ y(t) &= y(0) \end{aligned}\right\}$$

$$\vec{x}(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x(0) \\ y(0) \end{bmatrix}$$

$$\vec{x}(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \vec{x}(0)$$

By HW exercise soln to

$$\frac{d\vec{x}}{dt} = B\vec{x} \rightarrow x(t) = \boxed{e^{\lambda t}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}} \vec{x}(0)$$

$$\left[e^{tB}\right] = e^{\lambda t}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix}$$

BACK TO ORIG PROBLEM:

$$x(t) = S\left[e^{tB}\right]S^{-1}\vec{x}(0)$$

$$= \underbrace{e^{-2t}\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$e^{-2t}\begin{bmatrix} 1 & t \\ -2 & 1-2t \end{bmatrix}\begin{bmatrix} 3 \\ 8 \end{bmatrix} = e^{-2t}\begin{bmatrix} 3+8t \\ -6+8-16t \end{bmatrix}$$

$$= e^{-2t}\begin{bmatrix} 3+8t \\ 2-16t \end{bmatrix}$$

what if
$$\left\{\begin{aligned}\frac{dx}{dt} &= \lambda x + y\\ \frac{dy}{dt} &= \qquad \lambda y + z\\ \frac{dz}{dt} &= \qquad\qquad \lambda z\end{aligned}\right\} \quad \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\frac{d\vec{x}}{dt} = B\vec{x} \qquad B = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$\vec{x}(t) = \left[e^{tB}\right]\vec{x}(0) \qquad = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + \lambda I = A + \lambda I$$

ex:
$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \qquad \lambda I - B = \begin{bmatrix} \lambda-2 & -1 & 0 \\ 0 & \lambda-2 & -1 \\ 0 & 0 & \lambda-2 \end{bmatrix}$$

$$P_B(\lambda) = (\lambda-2)^3 \to \lambda = 2$$

$AM = 3$
$GM = 1$

$$\left\{\begin{aligned} A\vec{v}_1 &= \lambda\vec{v}_1 \\ A\vec{v}_2 &= \vec{v}_1 + \lambda\vec{v}_2 \\ A\vec{v}_3 &= \qquad \vec{v}_2 + \lambda\vec{v}_3 \end{aligned}\right\} \quad \vec{v}_1 \text{ e-vector}$$

$$(A - \lambda I)\vec{v}_2 = \vec{v}_1$$
$$(A-\lambda I)^2\vec{v}_2 = (A-\lambda I)\vec{v}_1 = \vec{0}$$

$$A\vec{v}_3 - \lambda I\vec{v}_3 = \vec{v}_2$$
$$(A-\lambda I)\vec{v}_3 = \vec{v}_2$$
$$(A-\lambda I)^2\vec{v}_3 = (A-\lambda I)\vec{v}_2 = \vec{v}_1$$
$$(A-\lambda I)^3\vec{v}_3 = (A-\lambda I)\vec{v}_1 = \vec{0}$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad [e^{tA}] = ?$$

$$\frac{dx}{dt} = y = c_1 t + c_2 \rightarrow x(t) = \frac{1}{2} c_1 t^2 + c_2 t + c_3$$

$$x(0) = c_3$$

$$\frac{dy}{dt} = z = c_1 \longrightarrow y(t) = c_1 t + c_2$$

$$y(0) = c_2$$

$$\frac{dz}{dt} = 0 \longrightarrow z(t) = c_1 = z(0)$$

$$\Rightarrow \begin{cases} x(t) = x(0) + t\,y(0) + \frac{1}{2}t^2 z(0) \\ y(t) = \qquad\quad y(0) + t\,z(0) \\ z(t) = \qquad\qquad\qquad z(0) \end{cases}$$

$$\vec{x}(t) = \underbrace{\begin{bmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}}_{[e^{tA}]} \begin{pmatrix} x(0) \\ y(0) \\ z(0) \end{pmatrix}$$

$$[e^{tB}] = e^{\lambda t} \begin{bmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

EASY NONLINEAR SYSTEM.

$$\begin{cases} \dfrac{dx}{dt} = x + y + 5 \\ \dfrac{dy}{dt} = -4x + y + 10 \end{cases} \qquad x(0) = 2, \quad y(0) = -1$$

$$* \quad \vec{x} = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \frac{d\vec{x}}{dt} = A\vec{x} + \vec{b}, \quad \vec{b} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 \\ -4 & 1 \end{bmatrix}$$

See Lecture #13 notes for details.

Contrast with $\begin{cases} \dfrac{dx}{dt} = x + y \\ \dfrac{dy}{dt} = -4x + y \end{cases} \quad \dfrac{d\vec{x}}{dt} = A\vec{x}$

EQUILIBRIUM at $(0, 0)$.

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* EQUILIBRIUM $\quad \dfrac{d\vec{x}}{dt} = \vec{0}$

$$A\vec{x} + \vec{b} = \vec{0} \qquad A\vec{x} = -\vec{b}$$

If A is invertible, then $\boxed{\vec{x} = -A^{-1}\vec{b}}$

$$A^{-1} = \frac{1}{5}\begin{bmatrix} 1 & -1 \\ 4 & 1 \end{bmatrix} \qquad \vec{x}_p = -\frac{1}{5}\begin{bmatrix} 1 & -1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 10 \end{bmatrix}$$

$$= -\begin{bmatrix} 1 & -1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -6 \end{bmatrix} = \vec{x}_p$$

THIS COINCIDES WITH EQUILIBRIUM INDICATED IN PPLANE.

FINALLY, NONLINEAR SYSTEMS, NONAUTONOMOUS.

Refer to Lecture Notes #13: UNDETERMINED COEFFICIENTS, VARIATION OF PARAMETERS