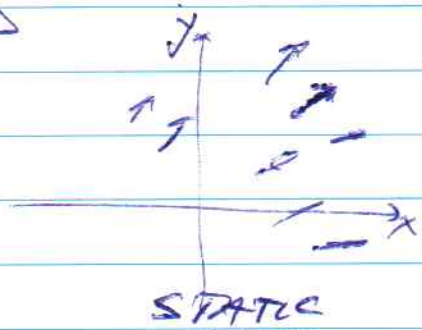
NONLINEAR SYSTEMS (2 VARIABLES)

$$\begin{cases} \dfrac{dx}{dt} = F(x,y,t) \\[2mm] \dfrac{dy}{dt} = G(x,y,t) \end{cases}$$ NONAUTONOMOUS (time-dependent)

$$\begin{cases} \dfrac{dx}{dt} = F(x,y) \\[2mm] \dfrac{dy}{dt} = G(x,y) \end{cases}$$



time indep. autonomous

STATIC

Population dynamics : LOTKA-VOLTERRA

Relative Growth Rates

$$\begin{cases} \dfrac{1}{x}\dfrac{dx}{dt} = a_1 + b_1 x + c_1 y \\[2mm] \dfrac{1}{y}\dfrac{dy}{dt} = a_2 + b_2 x + c_2 y \end{cases}$$

$$\frac{dx}{dt} = (a_1 + b_1 x + c_1 y)\,x = a_1 x + b_1 x^2 + c_1 x y = F(x,y)$$

$$\frac{dy}{dt} = (a_2 + b_2 x + c_2 y)\,y = a_2 y + b_2 x y + c_2 y^2 = G(x,y)$$

Example: $\begin{cases} \dfrac{dx}{dt} = x(6-2x-y) \\[2mm] \dfrac{dy}{dt} = y(4-x-y) \end{cases}$ competitive system

QUALITATIVE ANALYSIS — phase plane analysis

Key: NULLCLINES - HORIZONTAL and VERTICAL

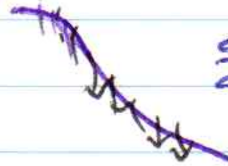
EQUILIBRIA

HORIZONTAL NULLCLINE (HNC) = curve along which vector field is purely horizontal.

$\frac{dy}{dt}=0$

$G(x,y)=0$

vertical NULLCLINE (VNC)
= curve along which vector field is purely vertical



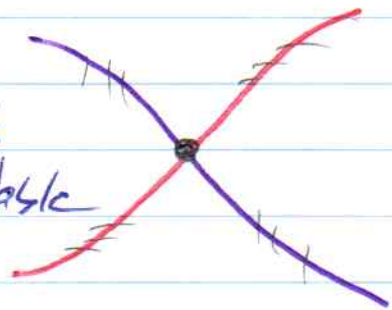
$\frac{dx}{dt}=0$

$F(x,y)=0$

EQUILIBRIA – POINT where vector field vanishes.

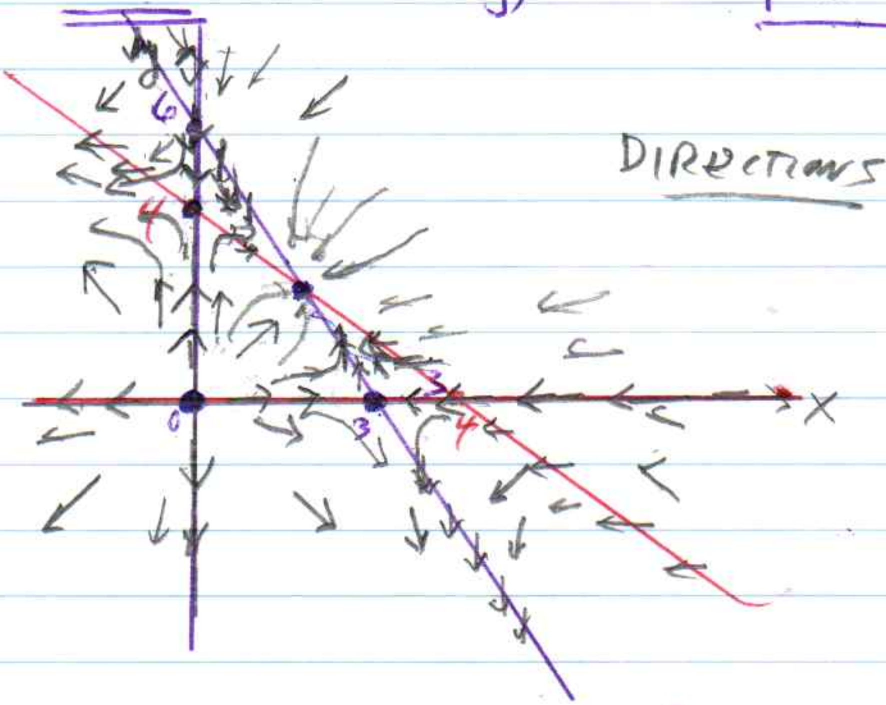
$$\begin{cases} \frac{dx}{dt} = F(x,y) \\ \frac{dy}{dt} = G(x,y) \end{cases}$$

F, G diff'able



HNC $y(4-x-y)=0 \Rightarrow \boxed{y=0}$ OR $\boxed{x+y=4}$

VNC $x(6-2x-y)=0 \Rightarrow \boxed{x=0}$ OR $\boxed{2x+y=6}$



$x=2$

$y=2$

stable $(2,2)$ SINK

$(3,0)$ saddle

unstable $(0,0)$ SOURCE

$(0,4)$ saddle

EQUILIBRIA ANALYSIS

LINEARIZATION NEAR AN EQUILIBRIUM

(x_0, y_0) EQUILIBRIUM

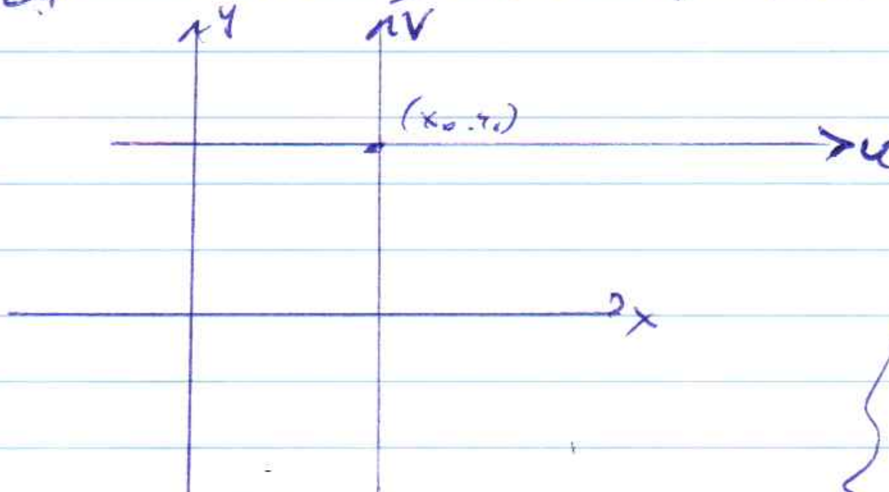
$F(x_0, y_0) = G(x_0, y_0) = 0$

21a

$$\begin{cases} \dfrac{dx}{dt} = F(x,y) \\ \dfrac{dy}{dt} = G(x,y) \end{cases}$$

$$\frac{dx}{dt} = F(x,y) \cong \cancel{F(x_0,y_0)}^{0} + F_x(x_0,y_0)(x-x_0) + F_y(x_0,y_0)(y-y_0)$$

$$\frac{dy}{dt} = G(x,y) \cong \cancel{G(x_0,y_0)}^{0} + G_x(x_0,y_0)(x-x_0) + G_y(x_0,y_0)(y-y_0)$$



$u = x - x_0$

$v = y - y_0$

$$\begin{cases} \dfrac{du}{dt} = \dfrac{dx}{dt} \\ \dfrac{dv}{dt} = \dfrac{dy}{dt} \end{cases} \qquad \vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\boxed{\begin{aligned} \frac{du}{dt} &\cong F_x(x_0,y_0)\, u + F_y(x_0,y_0)\, v \\ \frac{dv}{dt} &\cong G_x(x_0,y_0)\, u + G_y(x_0,y_0)\, v \end{aligned}} \qquad \vec{u} = \begin{bmatrix} u \\ v \end{bmatrix}$$

$$\frac{d\vec{u}}{dt} \cong \underbrace{\begin{bmatrix} F_x(x_0,y_0) & F_y(x_0,y_0) \\ G_x(x_0,y_0) & G_y(x_0,y_0) \end{bmatrix}}_{J} \vec{u} = \left[J(x_0,y_0) \right] \vec{u}$$

JACOBIAN MATRIX.

J

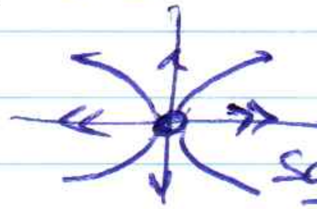
JACOBIAN ANALYSIS

In exaple $\frac{dx}{dt} = 6x - 2x^2 - xy = F(x,y)$

$\frac{dy}{dt} = 4y - xy - y^2 = G(x,y)$

$$J = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix} = \begin{bmatrix} 6-4x-y & -x \\ -y & 4-x-2y \end{bmatrix}$$

$$J(0,0) = \begin{bmatrix} 6 & 0 \\ 0 & 4 \end{bmatrix} \quad \lambda_1 = 6 \ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \lambda_2 = 4 \ \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



Source

$$J(2,2) = \begin{bmatrix} -4 & -2 \\ -2 & -2 \end{bmatrix} = A \qquad \lambda I - A = \begin{bmatrix} \lambda+4 & 2 \\ 2 & \lambda+2 \end{bmatrix}$$

$$p(\lambda) = \lambda^2 + 6\lambda + 4 = 0$$

$$\lambda = \frac{-6 \pm \sqrt{36-16}}{2} = -3 \pm \sqrt{5}$$

SINK $\qquad \lambda_1, \lambda_2 < 0$

$$J(3,0) = \begin{bmatrix} -6 & -3 \\ 0 & 1 \end{bmatrix} = A \qquad \lambda I - A = \begin{bmatrix} \lambda+6 & 3 \\ 0 & \lambda-1 \end{bmatrix}$$

$$p(\lambda) = (\lambda+6)(\lambda-1) = 0$$

saddle $\qquad \lambda_1 = \underset{-}{-6} < 0 \quad \lambda_2 = \underset{+}{1} > 0$

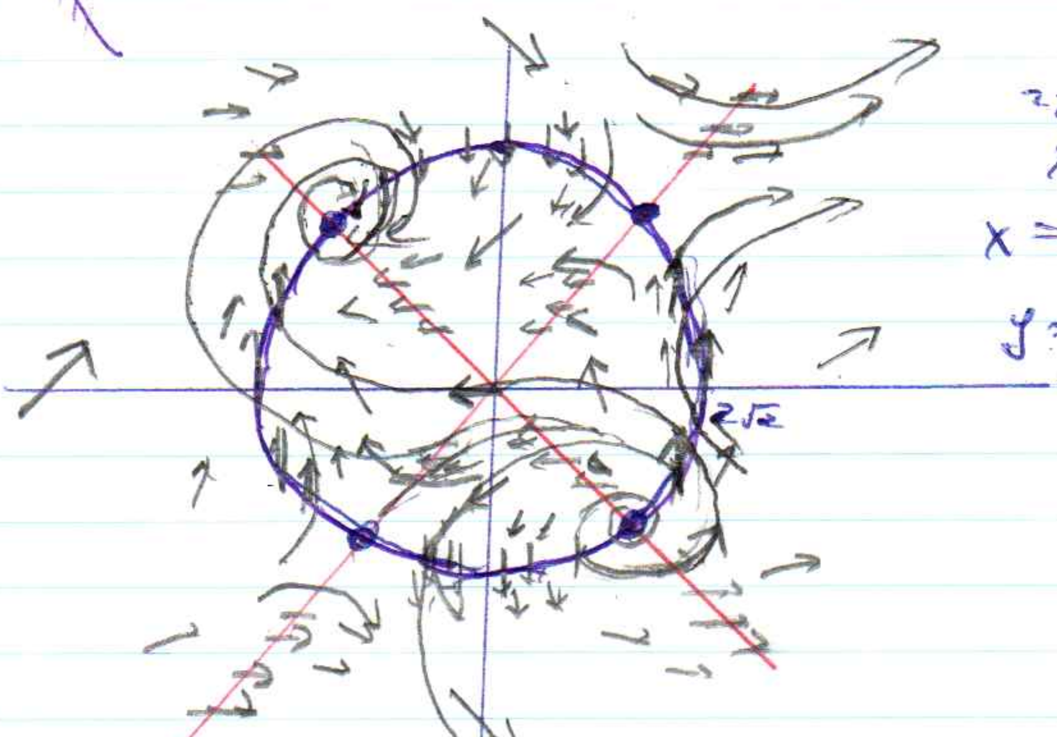
etc.

Example: $\left\{\begin{aligned} \frac{dx}{dt} &= x^2+y^2-8 = F(x,y) \\ \frac{dy}{dt} &= x^2-y^2 = G(x,y) \end{aligned}\right\}$

HNC: $x^2-y^2=(x-y)(x+y)=0 \rightarrow \boxed{y=x}$ OR $\boxed{y=-x}$

VNC: $x^2+y^2-8=0$

$x^2+y^2=8$



$2x^2=8$

$x^2=4$

$x=+2$ OR $x=-2$

$y=2$ or $y=-2$

4 EQUIL

$(2,2)$

$(2,-2)$

$(-2,2)$

$(-2,-2)$

$$J(x,y)=\begin{bmatrix} 2x & 2y \\ 2x & -2y \end{bmatrix}$$

$$J(2,2) = \begin{bmatrix} 4 & 4 \\ 4 & -4 \end{bmatrix} \quad \lambda I - A = \begin{bmatrix} \lambda-4 & -4 \\ -4 & \lambda+4 \end{bmatrix}$$

$$p(\lambda) = \lambda^2 - 32 \Rightarrow \lambda = \pm 4\sqrt{2}$$

$+, -$ SADDLE

$$J(2,-2) = \begin{pmatrix} 4 & -4 \\ 4 & 4 \end{pmatrix}$$

$$\lambda I - A = \begin{bmatrix} \lambda-4 & 4 \\ -4 & \lambda-4 \end{bmatrix}$$

$$p(\lambda) = (\lambda-4)^2 + 16 \rightarrow \lambda = 4 \pm 4i.$$

$Re(\lambda) > 0$ spiral out

$$J(-2,2) = \begin{bmatrix} -4 & -4 \\ -4 & 4 \end{bmatrix} \quad \lambda I - A = \begin{pmatrix} \lambda+4 & 4 \\ 4 & \lambda-4 \end{pmatrix}$$

$$\lambda^2 - 32 = 0 \quad \lambda = \pm 4\sqrt{2}$$

$+, -$ saddle

$$J(-2,2) = \begin{bmatrix} -4 & 4 \\ -4 & -4 \end{bmatrix} \quad \lambda I - A = \begin{bmatrix} \lambda+4 & -4 \\ 4 & \lambda+4 \end{bmatrix}$$

$$(\lambda+4)^2 + 16 = 0$$

$$\rightarrow \lambda = -4 \pm 4i$$

$Re(\lambda) < 0$

Inward spiral