ORDINARY DIFFERENTIAL EQUATIONS (ODE)

MODELING CHANGE

DISCRETE VS. CONTINUOUS.

Interest w/ compounding

Initial Amt. P_0, Annual Rate R
compound yearly.

$t=0 \quad P_0$

$t=1 \quad P_1 = P(1) = P_0 + P_0 R = P_0(1+R)$

$t=2 \quad P_2 = P(2) = P_1(1+R) = P_0(1+R)^2$

$\vdots$

t yrs. $\quad P_t = P(t) = P_0(1+R)^t$

[Compound monthly] $\quad R/12$ = monthly RATE

1 month $\quad P_0(1+R/12)$

2 month $\quad P_0(1+R/12)^2$

$\vdots$

1 yr = 12 months $\quad P_0\left(1+\frac{R}{12}\right)^{12}$

t yrs $\quad P(t) = P_0\left(1+\frac{R}{12}\right)^{12t}$

compound n times/yr. $\quad P(t) = P_0\left(1+\frac{R}{n}\right)^{nt}$

$$P(t) = P_0 \lim_{n\to\infty}\left(1+\frac{R}{n}\right)^{nt} = P_0\Bigg[\overbrace{\lim_{n\to\infty}\left(1+\frac{R}{n}\right)^{n}}^{e^R}\Bigg]^{t} = P_0 e^{Rt}$$

continuous compounding.

ALTERNATIVE - ODE.

GROWTH RATES

Natural Growth Model. $P(t)$

$$\boxed{\frac{dP}{dt} = RP} \ast \quad \text{1st order ODE}$$

Relative Growth Rate

$$\frac{dP/dt}{P} = \boxed{\frac{1}{P}\frac{dP}{dt} = R}$$ constant Rel. Growth Rate.

Initial Condition (IC) $P(0) = P_0$ *

⇒ Initial Value Problem (IVP) = ODE + IC

ODE - 1st ORDER

$$\frac{dP}{dt} = F(t, P) \quad \text{in general}$$

Special Cases $\frac{dP}{dt} = at$ $P(0) = P_0$

$$P(t) = \frac{1}{2}at^2 + C$$

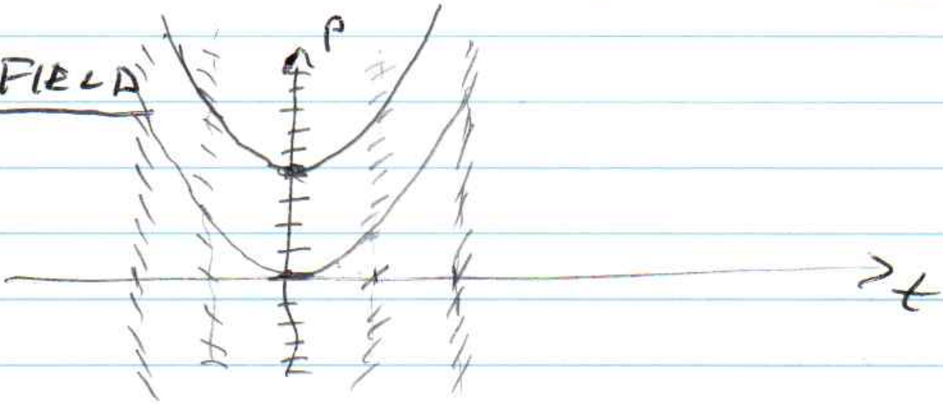
$$P(0) = 0 + C = C = P_0$$

$$\Rightarrow \boxed{P(t) = \frac{1}{2}at^2 + P_0}$$

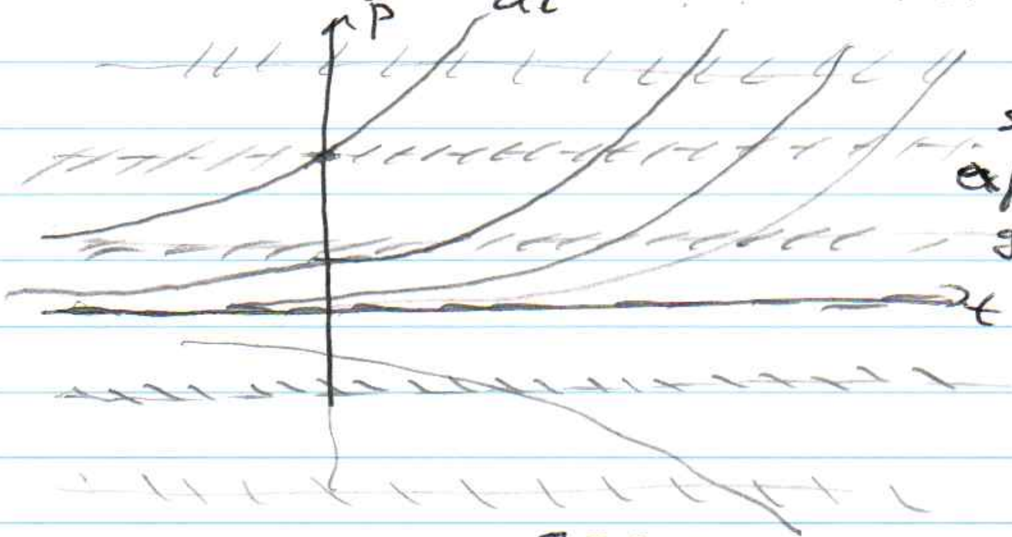
$$\frac{dP}{dt} = 2t \qquad P(0) = 5$$

$$P(t) = t^2 + C \Rightarrow \boxed{P(t) = t^2 + 5}$$

SLOPE FIELD



NATURAL GROWTH: $\frac{dP}{dt} = RP \qquad P(0) = P_0$



suggests exponential growth.

SEPARATION OF VARIABLES

$$\frac{dP}{dt} = F(t, P)$$

$$\frac{dP}{dt} = RP$$

$$\frac{1}{P}\frac{dP}{dt} = R$$

$$\frac{d}{dt}(\ln P) = R$$

$$\ln P = Rt + C$$

$$e \qquad e$$

$$P(t) = Ae^{Rt}$$

$$P(0) = Ae^0 = A = P_0$$

$$\Rightarrow \boxed{P(t) = P_0 e^{Rt}}$$

or ... $\frac{dP}{dt} = RP \qquad P(0) = P_0$

Separation of VARIABLES

$$\int \frac{dP}{P} = \int R\,dt$$

$$e^{\ln|P|} = e^{Rt + C}$$

$$P(t) = Ae^{Rt} \qquad P(0) = \underline{A = P_0}$$

$$\Rightarrow \boxed{P(t) = P_0 e^{Rt}}$$

GROWTH IN A LIMITED ENVIRONMENT.

$$\boxed{\frac{1}{P}\frac{dP}{dt} = R\left(1 - \frac{P}{L}\right)}$$

L = CARRYING CAPACITY

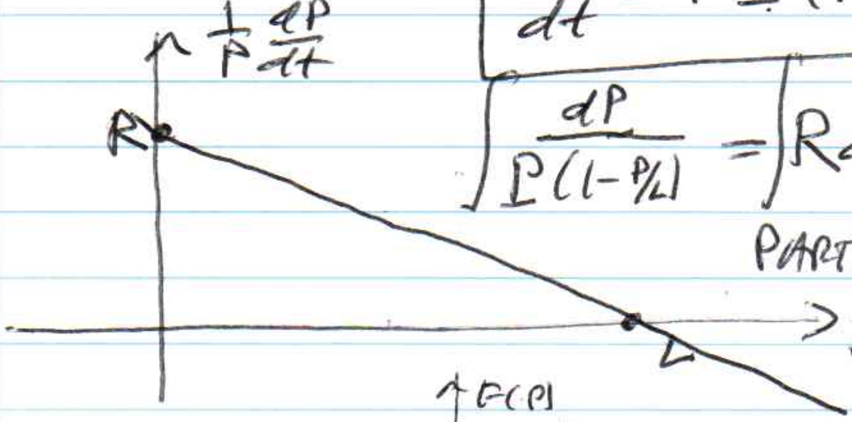
<u>LOGISTIC MODEL</u>

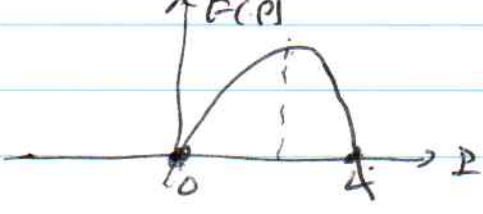
$$\boxed{\frac{dP}{dt} = RP\left(1 - \frac{P}{L}\right)} = F(P)$$

$$\int \frac{dP}{P(1 - P/L)} = \int R\,dt$$

PARTIAL FRACTIONS

solve for P.





SLOPE FIELD — LOGISTIC MODEL

$x(t)$

Def'n: An nth order Linear ORDINARY DIFFERENTIAL EQUATION is an ODE of form:

$$\boxed{\frac{d^n x}{dt^n} + p_{n-1}(t)\frac{d^{n-1}x}{dt^{n-1}} + \cdots + p_1(t)\frac{dx}{dt} + p_0(t)\cdot x = g(t)}$$

functions of t

Note: much easier if all $p_k(t)$ constant

$g(t) = 0 \longleftrightarrow$ homogenous

$g(t) \neq 0 \longleftrightarrow$ inhomogeneous.

LINEAR? "preserves scaling and adding"

$\frac{d}{dt} = D =$ "take the derivative"

$$\left.\begin{aligned} \frac{d}{dt}\left[f(t)+g(t)\right] &= f'(t)+g'(t) \\ \frac{d}{dt}\left[c f(t)\right] &= c f'(t) \end{aligned}\right\} \frac{d}{dt} \text{ acts linearly.}$$

$$\frac{d}{dt}\left[c_1 x_1(t) + c_2 x_2(t)\right] = c_1 x_1'(t) + c_2 x_2'(t)$$

$D = \frac{d}{dt}$: functions $\longrightarrow$ functions
"take the derivative"

composition of linear function (operators) is also linear.

$D \circ D = \frac{d^2}{dt}$ acts linearly.

multiplication by a function acts linearly.

$g(t)\left[c_1 f_1(t) + c_2 f_2(t)\right]$ [Just the DISTRIBUTIVE LAW!]

$= c_1 g(t) f_1(t) + c_2 g(t) f_2(t)$

Algebraic combinations of linear operators (sums, differences, etc) act linearly.

$D = \frac{d}{dt}$ An nth order linear ODE takes the form:

$$\underbrace{\left[D^n + p_{n-1}(t) D^{n-1} + \cdots + p_1(t) D + p_0(t) I\right]}_{T} x(t) = g(t)$$

$T(x(t)) = g(t)$

T a <u>linear operator</u>

LINEARITY METHOD (superposition).

Solve $T(x(t)) = g(t)$ $(*)$

Suppose $x_p(t)$ is just one particular solution to $(*)$.

Let $x(t)$ be any other solution to $(*)$.

Then $x(t) - x_p(t)$ solves the homogenous ODE.

$$T\left(x(t) - x_p(t)\right) = T(x(t)) - T(x_p(t))$$
$$= g(t) - g(t) = 0$$

If $x_h(t)$ can be found that yields all solutions to homogeneous eqn

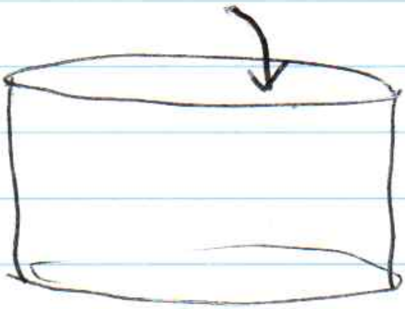
$$\boxed{T(x_h(t)) = 0}$$

then $x(t) - x_p(t) = x_h(t)$

$\therefore$ $x(t) = x_h(t) + x_p(t)$

method: ① Find all homog soln $x_h(t)$
② Find any particular soln $x_p(t)$
③ general soln is $x(t) = x_h(t) + x_p(t)$
④ Use Initial conditions to then get unique solution to IVP.

MIXING PROBLEM



Pour in salty water at RATE of 10 ℓ/hr with concentration 50 gm/liter

1000 ℓ of a salty mixture

STIR constantly.

Initial concentration 30 gm/ℓ

Let excess spill out.

Q: What is concentration after 20 hours?

Let $x(t)$ = amt of salt (in gm) in VAT at any given time.

$$\frac{dx}{dt} = (\text{Rate In}) - (\text{Rate Out})$$

$$\frac{dx}{dt} = \left(\frac{10\,\ell}{hr}\right)\left(\frac{50\,gm}{\ell}\right) - \left(\frac{10\,\ell}{hr}\right)\left(\frac{X\,gm}{1000\,\ell}\right)$$

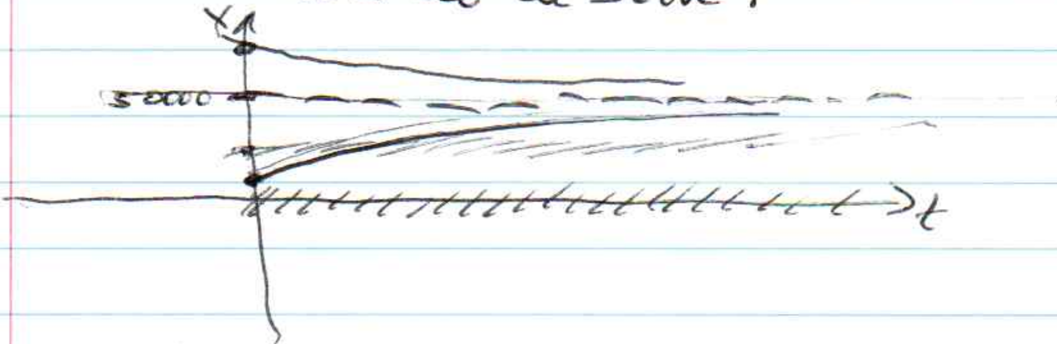
$$\boxed{\frac{dx}{dt} = 500 - .01X}$$

IVP

$$X(0) = \left(30\,gm/\ell\right)\left(1000\,\ell\right) = 30000\,gm$$

$$\boxed{\frac{dx}{dt} = 500 - .01x} \qquad x(0) = 30000 \qquad \underline{\underline{IVP}}$$

How do we solve?



EQUILIBRIUM SOLUTION

$$\frac{dx}{dt} = 0 \text{ when } 500 - .01x = 0$$

$$.01x = 500 \qquad \boxed{x = 50000}$$

$\boxed{\text{I}}$ Separation of VARIABLES

$$\frac{-.01\,dx}{500 - .01x} = -.01\,dt$$

$$\int \frac{-.01\,dx}{500 - .01x} = \int -.01\,dt$$

$$e^{\ln|500 - .01x|} = e^{-.01t + C}$$

$$500 - .01x = Ae^{-.01t}$$

$$t = 0 \Leftrightarrow x = 30000 \qquad 200 = A$$

$$500 - .01x = 200e^{-.01t}$$

$$.01x = 500 - 200e^{-.01t}$$

$$\boxed{x(t) = 50000 - 20000e^{-.01t}}$$

II LINEARITY METHOD

$$\boxed{\frac{dx}{dt} + .01x = 500} \quad x(0) = 30000$$

① HOMOG $\frac{dx}{dt} + .01x = 0$

$$\frac{dx}{dt} = -.01x$$

$$\int \frac{dx}{x} = -\int .01\,dt$$

Parameter ↓

$$e^{\ln x} = e^{-.01t + C} \Rightarrow \boxed{x_h(t) = Ae^{-.01t}}$$

② PART. SOLN.

$$x_p(t) = 50000$$ ✓ (by inspection) (we'll soon learn other methods)

③ $x(t) = Ae^{-.01t} + 50000$ (general solution)

④ $x(0) = A + 50000 = 30000$ (use INITIAL CONDITION)

$$\therefore A = -20000$$

$$\Rightarrow \boxed{x(t) = 50000 - 20000e^{-.01t}}$$

20 hrs $\frac{x(20)}{1000\ell} = \frac{50000 - 20000e^{-.2}}{1000\ell}$

$$\left[= 33.6254 \ \text{gm}/\ell \right]$$

$\boxed{III}$ Integrating FACTOR. $u = u(t)$

(1st ORDER)

$$\frac{dx}{dt} + p(t)\, x = q(t)$$

Can always be solved

$$\frac{dx}{dt} + p x = q$$

$$\underbrace{u \frac{dx}{dt} + (up) x}_{} = q u$$

$$\frac{d}{dt}(ux) = u \frac{dx}{dt} + x \frac{du}{dt} = q u$$

Product Rule

$$\frac{d}{dt}[ux] = u\frac{dx}{dt} + x\frac{du}{dt}$$

$$\boxed{\frac{du}{dt} = up}$$

separable

Solve: $\int \frac{du}{u} = \int p\, dt \qquad \ln u = \int p\, dt$

$$e^{\ln u} = e^{\int p\, dt}$$

$$\boxed{u(t) = e^{\int p(t)\, dt}}$$ integrating factor

$$\frac{dx}{dt} + .01x = 500$$

$p(t) = .01 \qquad e^{\int p\, dt} = \boxed{e^{.01t}}$ Integrating factor

$$\underbrace{e^{.01t} \frac{dx}{dt} + .01x \cdot e^{.01t}}_{} = 500 e^{.01t}$$

$$\frac{d}{dt}\left[e^{.01t} x\right] = 500 e^{.01t}$$

$$e^{.01t}x = 5000e^{.01t} + C$$

$$x(t) = 50000 + Ce^{-.01t}$$

$$IC \Rightarrow \quad C = -20000$$

$$\Rightarrow x(t) = 50000 - 20000e^{-.01t}$$