RECAP:

SEP'N of VARIABLE, LINEAR ODE'S

Integrating Factor

$\rightarrow$ 1st order ODE's: $\frac{dx}{dt} + p(t)\,x = g(t)$

LINEAR EQNS + LINEARITY.

$\Rightarrow$ Homog + PART = general.

PARTICULAR SOLUTION: ① Undetermined Coefficients
② VARIATION of Parameters.

LINEARITY:

(*) $[p(D)]\,x(t) = c_1 g_1(t) + c_2 g_2(t)$

IDEA! Solve $[p(D)]\,x(t) = g_1(t) \Rightarrow x_1(t)$
solve $[p(D)]\,x(t) = g_2(t) \Rightarrow x_2(t)$

$x(t) = c_1 x_1(t) + c_2 x_2(t)$ solves (*)

Reason:

$[p(D)]\,x(t) = [p(D)](c_1 x_1(t) + c_2 x_2(t))$

$= c_1 [p(D)]\,x_1(t) + c_2 [p(D)]\,x_2(t)$

$= c_1 g_1(t) + c_2 g_2(t)$

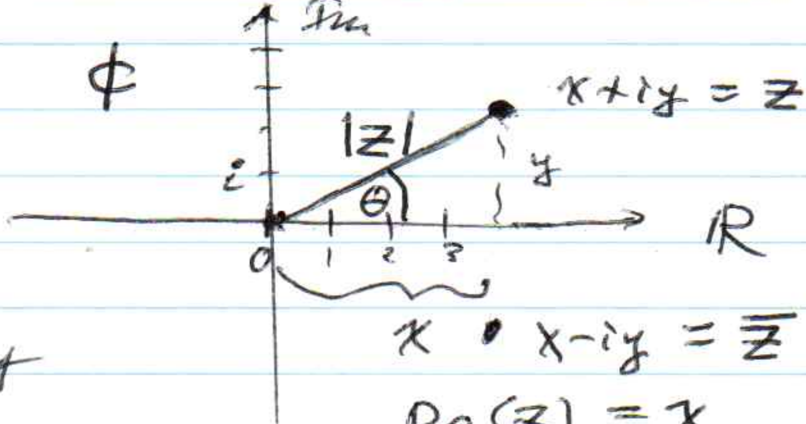
Ex: $\ddot{x} + 5\dot{x} + 4x = 3\sin 2t + 4t^2$

DO IT IN PIECES.

[NOTE: Here $p(D) = D^2 + 5D + 4I$]

Complex numbers and complex functions

BASICS $\mathbb{C}$



imaginary

complex

$Re(z) = x$

$Im(z) = y$

$$i^2 = -1$$

$$z^2 + 1 = 0$$

Addition:

$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$$

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2)$$
$$= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$$

Scaling by Real number

$$t(z_1 + z_2) = tz_1 + tz_2$$

Complex conjugates $\bar{z} = x - iy$

$$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$$

$$\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$ *

$|z|$ = modulus (z)

* check quadrant

θ = argument (z) $\tan\theta = \frac{y}{x}$

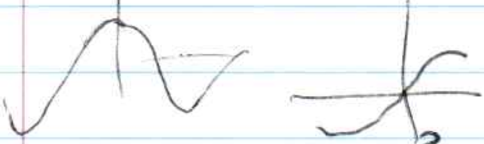
Euler's Formula: $\boxed{e^{it} = \cos t + i \sin t}$

Euler's Identity: $\boxed{e^{i\pi} = -1}$

MACLAURIN SERIES

$$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots + \frac{t^n}{n!} + \cdots$$

$$\cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots \qquad \text{(even)}$$



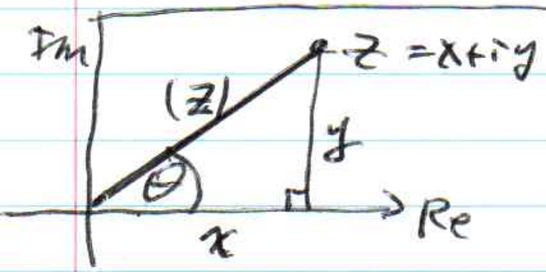
Absolutely convergent

$$\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots \qquad \text{(odd)}$$

$$e^{it} = 1 + it + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \frac{(it)^4}{4!} + \cdots$$

$$= \left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots\right) + i\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots\right)$$

$$e^{it} = \cos t + i \sin t \quad \checkmark$$



$$|z|^2 = x^2 + y^2$$

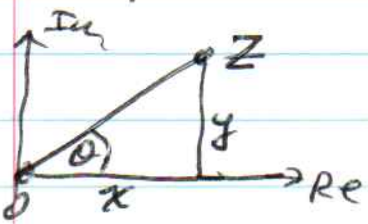
$$|z| = \sqrt{x^2 + y^2}$$

$$x = |z| \cos\theta \qquad y = |z| \sin\theta$$

$$z = \underbrace{x + iy}_{\text{Rectangular form}} = |z|\cos\theta + i|z|\sin\theta$$

$$= |z|(\cos\theta + i\sin\theta) = \underbrace{|z|e^{i\theta}}_{\text{polar form}}$$

Complex multiplication – polar form.



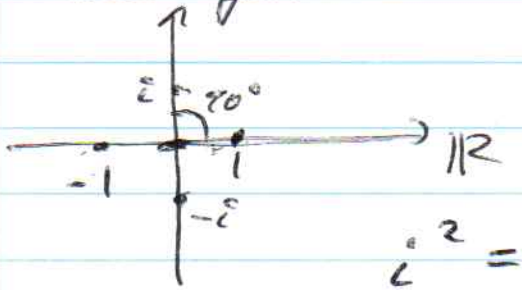
$$z_1 = |z_1| e^{i\theta_1} \qquad z_2 = |z_2| e^{i\theta_2}$$

$$z_1 z_2 = |z_1||z_2| e^{i\theta_1} e^{i\theta_2}$$

$$= \underbrace{|z_1||z_2|}_{|z_1 z_2|} e^{i(\theta_1 + \theta_2)}$$

$\arg(z_1 z_2)$

multiply moduli
add arguments



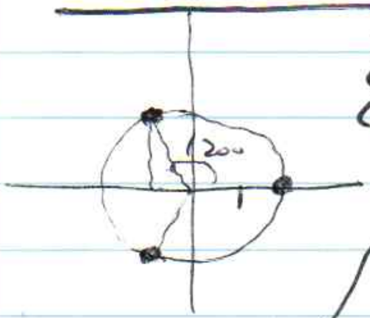
$$i = e^{i\pi/2}$$

$$i^2 = i \cdot i = e^{i\pi} = -1$$

$$i^3 = -i \qquad i^4 = 1$$

Roots of Unity

$$\left\{ 1, -\frac{1}{2} + i\frac{\sqrt{3}}{2}, -\frac{1}{2} - i\frac{\sqrt{3}}{2} \right\}$$



$$z^3 - 1 = 0$$

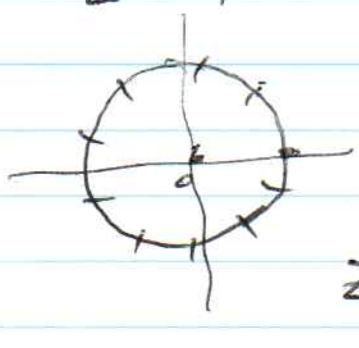
$$(z-1)(z^2 + z + 1) = 0$$

$z = 1$

$$z = \frac{-1 \pm \sqrt{1-4}}{2}$$

$$= -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

$$z^3 = 1$$



$$z^n = 1 \quad \text{Roots: } \left\{ e^{i\frac{2\pi k}{n}} \right\}_{k=0}^{n-1}$$

TRIG APPLICATION: Sum of angle formulas

$$e^{i(\theta+\phi)} = \cos(\theta+\phi) + i\sin(\theta+\phi)$$

$$\parallel$$

$$e^{i\theta}e^{i\phi} = (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)$$

$$= (\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\cos\theta\sin\phi + \sin\theta\cos\phi)$$

$$\left\{\begin{array}{l} \cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi \\ \sin(\theta+\phi) = \cos\theta\sin\phi + \sin\theta\cos\phi \end{array}\right\}$$

Calculus Application

$$\int e^{2t}\sin 3t\,dt$$ Integration by Parts 2X

Solve the integral.

$$\boxed{\int e^{at}\cos bt\,dt} + i\boxed{\int e^{at}\sin bt\,dt}$$

$$= \int e^{at}(\cos bt + i\sin bt)\,dt$$

$$= \int e^{at}e^{ibt}\,dt = \int e^{(a+ib)t}\,dt$$

$$\boxed{\int e^{rt}dt = \frac{e^{rt}}{r}}$$

$$= \frac{e^{(a+ib)t}}{a+ib}\left(\frac{a-ib}{a-ib}\right)$$

$$= \frac{(a-ib)(\cos bt + i\sin bt)}{a^2+b^2}$$

$$z\bar{z}$$
$$= (x+iy)(x-iy)$$
$$= x^2+y^2 = |z|^2$$

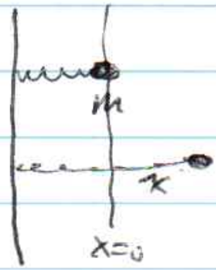
$$\frac{1}{a^2+b^2}\left[(a\cos bt + b\sin bt) + i(a\sin bt - b\cos bt)\right]$$

$$\Rightarrow \int e^{at}\cos bt\,dt = \frac{1}{a^2+b^2}(a\cos bt + b\sin bt) + c$$

$$\int e^{at}\sin bt\,dt = \frac{1}{a^2+b^2}(a\sin bt - b\cos bt) + c$$

ODE Application

I Solve $\boxed{\ddot{x} + \omega^2 x = 0}$



$ma = F = -kx \qquad a = -\frac{k}{m}x = \ddot{x}$

$\ddot{x} + \frac{k}{m}x = 0 \qquad \ddot{x} + \omega^2 x = 0$

Homog solns: characteristic polynom.

$x = e^{rt} \qquad \dot{x} = re^{rt} \qquad \ddot{x} = r^2 e^{rt}$

$(r^2 + \omega^2)e^{rt} = 0$

$p(r) = r^2 + \omega^2 = 0 \qquad \boxed{r = \pm i\omega}$

$\text{Span}\{e^{i\omega t}, e^{-i\omega t}\} \rightarrow$ homog solns.

$x(t) = c_1 e^{i\omega t} + c_2 e^{-i\omega t} \qquad c_1, c_2 \in \mathbb{C}$

$= c_1(\cos\omega t + i\sin\omega t) + c_2(\cos\omega t - i\sin\omega t)$

$= (c_1 + c_2)\cos\omega t + i(c_1 - c_2)\sin\omega t$

$= b_1\cos\omega t + b_2\sin\omega t$ Get Real!

$b_1, b_2 \in \mathbb{R}$

Note: If r is a double root (Resonance)

$$\boxed{\text{RRF}} \quad x_p(t) = \frac{a t e^{rt}}{p'(r)}$$

$$\ldots \quad x_p(t) = \frac{a t^2 e^{rt}}{p''(r)}$$

etc.

HARMONIC RESPONSE

Solve $\ddot{x} + k^2 x = a \cos \omega t$ (A)

PARTICULAR SOLUTION

Solve $\ddot{y} + k^2 y = a \sin \omega t$ (B)

COMPLEX REPLACEMENT

$$z = x + iy$$
$$\dot{z} = \dot{x} + i\dot{y}$$
$$\ddot{z} = \ddot{x} + i\ddot{y}$$

$$\begin{aligned}\ddot{z} + k^2 z &= (\ddot{x} + k^2 x) + i(\ddot{y} + k^2 y) \\ &= (a \cos \omega t) + i(a \sin \omega t) \\ &= a(\cos \omega t + i \sin \omega t) = a e^{i\omega t}\end{aligned}$$

Solve $\boxed{\ddot{z} + k^2 z = a e^{i\omega t}}$

EASY w/ ERF

$p(r) = r^2 + k^2$

characteristic polynomial

Re z solves (A)

Im z solve (B)

$\text{Span}\{\cos\omega t, \sin\omega t\}$

Sinusoidal Response

$\Pi(x(t)) = a\cos\omega t$

$$\left[\ \ddot{x} + 5\dot{x} + 4x = 5\sin 3t\ \right]$$

SIDETRACK: $\boxed{\Pi(x(t)) = ae^{rt}}$ exponential Input.

$$* \left[\ddot{x} + 5\dot{x} + 4x = 5e^{3t}\right]$$

Method of undet. coeffs.

Try soln of for $x(t) = ce^{rt}$

Plug into $\Pi(x(t)) = ae^{rt}$

$$c\,p(r) \cdot \cancel{e^{rt}} = a\cancel{e^{rt}}$$

$$\Rightarrow \boxed{c = \frac{a}{p(r)}}$$

$\Rightarrow$ EXPONENTIAL RESPONSE FORMULA.

$$\boxed{x_p(t) = \frac{ae^{rt}}{p(r)}} \quad \text{ERF}$$

r not a char. Root

$* \; p(r) = r^2 + 5r + 4$

$$x_p(t) = \frac{5e^{3t}}{p(3)} = \frac{5e^{3t}}{9+15+4} = \frac{5}{28}e^{3t}$$

For $\ddot{z} + k^2 z = a e^{i\omega t}$

$$z(t) = \frac{a e^{i\omega t}}{p(i\omega)} \qquad r = i\omega$$

$$= \frac{a e^{i\omega t}}{k^2 - \omega^2} \qquad p(r) = r^2 + k^2$$

$$p(i\omega) = (i\omega)^2 + k^2 = k^2 - \omega^2$$

$$\frac{a}{k^2-\omega^2}\left[\cos \omega t + i \sin \omega t\right]$$

$\mathrm{Re}(z) = x = \frac{a}{k^2-\omega^2} \cos \omega t$ solves (A)

$\mathrm{Im}(z) = y = \frac{a}{k^2-\omega^2} \sin \omega t$ solves (B)

Find part. soln to $\ddot{x} + 5\dot{x} + 4x = 3\cos 2t$

$$p(r) = r^2 + 5r + 4$$

<u>cx Repl</u>: $\ddot{z} + 5\dot{z} + 4z = 3e^{2it}$

$$\text{ERF} \Rightarrow z = \frac{3e^{2it}}{p(2i)} = \boxed{\frac{3e^{2it}}{10i}}$$

$$p(2i) = -4 + 10i + 4 = 10i$$

$$z = \frac{3(\cos 2t + i \sin 2t)}{10i} \cdot \frac{i}{i} = -\frac{3}{10}\left(-\sin 2t + i\cos 2t\right)$$

Real Part $= \frac{3}{10}\sin 2t = x_p(t)$

Alternatively

$$\frac{3e^{2it}}{ae^{i\phi}} = \frac{3}{a}e^{i(2t-\phi)}$$

$$= \boxed{\frac{3}{a}\Big[\cos(2t-\phi)}+ i\sin(2t-\phi)\Big]$$

phase angle

ADDENDUM

Here's a better example:

Find a particular solution to $\ddot{x} + 5\dot{x} + 4x = 4\sin 3t$

characteristic polynomial is $p(r) = r^2 + 5r + 4$.

[Roots are $r=-1$, $r=-4$, so $x_h(t) = c_1 e^{-t} + c_2 e^{-4t}$.]

For particular solution, use complex replacement and ERF.

Instead, solve $\ddot{z} + 5\dot{z} + 4z = 4e^{3it}$

then take imaginary part.

ERF $\Rightarrow z_p = \dfrac{4e^{3it}}{p(3i)}$ $\quad \left(\begin{aligned} p(3i) &= (3i)^2 + 5(3i) + 4 \\ &= -9 + 15i + 4 = -5 + 15i \end{aligned}\right)$

$$\Rightarrow z_p = \frac{4e^{3it}}{-5+15i}$$

$$= \frac{4}{5}\left(\frac{e^{3it}}{-1+3i}\right)$$

there are two good ways to proceed from here

$\boxed{\text{I}}$ SPLIT INTO REAL + IMAG. parts:

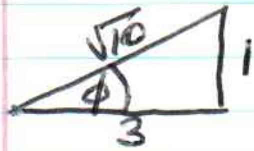
$$z_p = \frac{4}{5}\left(\frac{\cos 3t + i\sin 3t}{-1+3i}\right)\left(\frac{-1-3i}{-1-3i}\right) = \frac{2}{25}\left[\begin{aligned}&(-\cos 3t + 3\sin 3t)\\ &-i(3\cos 3t + \sin 3t)\end{aligned}\right]$$

$\Longrightarrow$

We only want the imaginary part, so

$$x_p(t) = -\frac{2}{25}\left(3\cos 3t + \sin 3t\right)$$

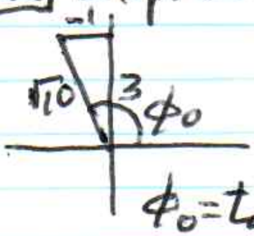
We can combine terms to get a cosine with phase angle



$$\phi = \tan^{-1}\left(\frac{1}{3}\right)$$

$$\Longrightarrow \boxed{x_p(t) = -\frac{2}{25}\sqrt{10}\cos(3t-\phi)} \quad \text{(A)}$$

$\boxed{\text{II}}$ Express denominator in polar form



$$-1+3i = \sqrt{10}\, e^{i\phi_0}$$

$$\text{So } z_p = \frac{4}{5}\left(\frac{e^{3it}}{\sqrt{10}\, e^{i\phi_0}}\right)$$

$$\phi_0 = \tan^{-1}\left(-\tfrac{1}{3}\right)$$

(Note $\phi_0 = \phi + \pi/2$ from above)

$$= \frac{2\sqrt{10}}{25} e^{i(3t-\phi_0)}$$

The imaginary part is then

$$\boxed{x_p(t) = \frac{2\sqrt{10}}{25}\sin(3t-\phi_0)} \quad \text{(B)}$$

This seems different than (A) above, but if we relate ϕ_0 and ϕ via $\phi_0 = \phi + \pi/2$, we get

$$x_p(t) = \frac{2\sqrt{10}}{25}\sin\left((3t-\phi) - \pi/2\right)$$

$$= -\frac{2\sqrt{10}}{25}\cos(3t-\phi) \quad \text{as above.}$$

Note: Input amplitude was 4, and response amplitude is $\frac{2\sqrt{10}}{25}$.

$$\text{Ratio} = \frac{\text{Response ampl.}}{\text{Input ampl.}} = \frac{2\sqrt{10}/25}{4} = \frac{\sqrt{10}}{50} = \underline{\text{GAIN}}$$