Autonomous 1st ORDER ODES

1st ORDER: $\frac{dx}{dt} = F(t, x)$

$\boxed{\text{Autonomous}}$: $\frac{dx}{dt} = F(x)$

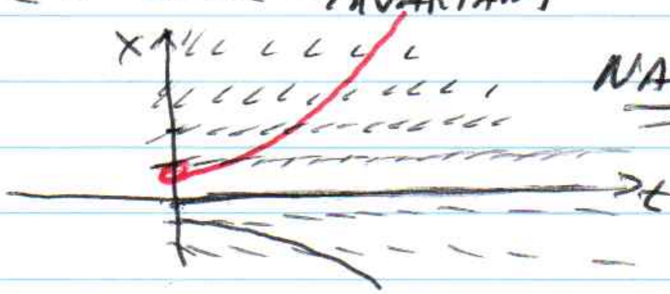
Natural Growth: $\frac{dx}{dt} = kx$, $x(0) = x_0$

$\Rightarrow x(t) = x_0 e^{kt}$

Logistic Model: $\frac{dx}{dt} = kx\left(1 - \frac{x}{L}\right) = F(x)$

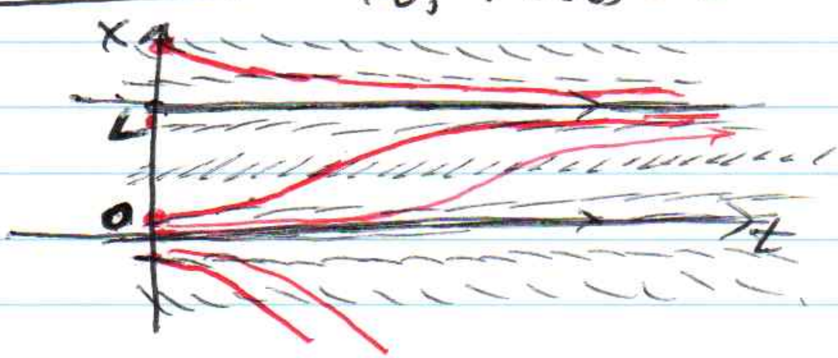
Autonomous $\leftrightarrow$ time-invariant

SLOPE FIELD



NATURAL

EQUILIBRIUM: x_0, $F(x_0) = 0$ $x(t) = x_0$



LOGISTIC

phase LINE

EQUILIBRIA

Def'ns: AN EQUILIBRIUM is STABLE if all nearby sol'ns attracted to the EQUILIBRIUM

" " UNSTABLE ...... Repelled away ....

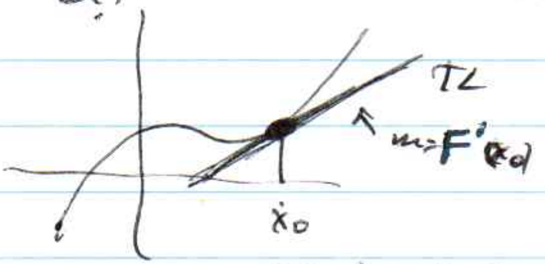
Semistable if combination of two.

DERIVATIVE TEST FOR STABILITY.

$\frac{dx}{dt} = F(x)$ F(x) differentiable

x near x_0

$$\frac{dx}{dt} = F(x) \cong F(x_0) + F'(x_0)(x - x_0)$$



$x - x_0$ small

$u = x - x_0$

$\frac{du}{dt} = \frac{dx}{dt}$

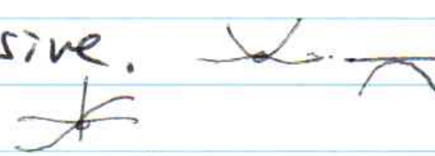
$F(x_0) = 0$ AT EQ

$$\boxed{\frac{du}{dt} \cong F'(x_0)\, u}$$ Approx. Nat. growth.

$F'(x_0) > 0 \iff$ exp. growth. $\iff$ unstable.

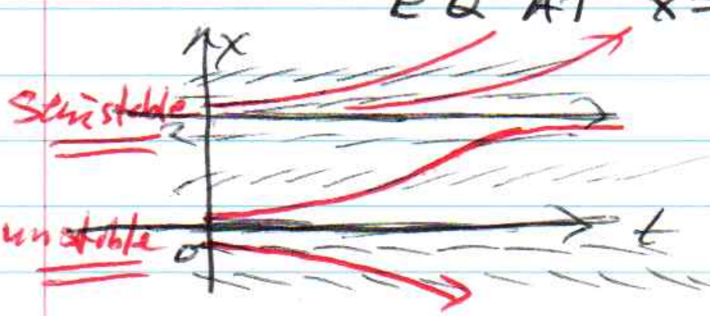
$F'(x_0) < 0 \iff$ exp decay $\iff$ stable

$F'(x_0) = 0 \iff$ Inconclusive.



Ex: $\frac{dx}{dt} = x(x-2)^2 = F(x)$

EQ AT $x = 0$ OR $x = 2$



$F(x) = x(x^2 - 4x + 4)$
$= x^3 - 4x^2 + 4x$

$F'(x) = 3x^2 - 8x + 4$

$F'(0) = 4$ unstable

$F'(2) = 12 - 16 + 4 = 0 \rightarrow$ semistable

$$\int \frac{dx}{x(x-2)^2} = \int dt$$ PARTIAL FRACTIONS $\Rightarrow x(t)$

Ref: Lecture notes #3 for LOGISTIC SOLUTION.

1st ORDER LINEAR, Inhomogeneous ODES.

$$\frac{dx}{dt} + p(t)\, x = g(t)$$

Integrating Factor $\boxed{e^{\int p(t)dt}}$

Ex: $\frac{dx}{dt} + tx = 2t$ $\quad p(t) = t$

$$\boxed{e^{\frac{1}{2}t^2}}$$

$$\underbrace{e^{\frac{1}{2}t^2}\frac{dx}{dt} + te^{\frac{1}{2}t^2}x}_{\frac{d}{dt}\left[e^{\frac{1}{2}t^2}x\right]} = 2te^{\frac{1}{2}t^2}$$

$$\frac{d}{dt}\left[e^{\frac{1}{2}t^2}x\right] = 2te^{\frac{1}{2}t^2} = 2\frac{d}{dt}\left(e^{\frac{1}{2}t^2}\right)$$

$$e^{\frac{1}{2}t^2}x = 2e^{\frac{1}{2}t^2} + C$$

$$x(t) = 2 + Ce^{-\frac{1}{2}t^2}$$

$$IC \rightarrow C \Rightarrow \boxed{x(t)}$$

LINEARITY: ① Homog ② PART ③ General

① $\frac{dx}{dt} + tx = 0 \quad \frac{dx}{dt} = -tx$

$$\int \frac{dx}{x} = \int -t\,dt \Rightarrow \ln|x| = -\frac{1}{2}t^2 + C'$$

$$x(t) = \boxed{Ae^{-\frac{1}{2}t^2} = x_h(t)}$$

② PARTICULAR $\frac{dx}{dt} + tx = 2t$

Try $x(t) = At + B$ undet coeffs.

$\frac{dx}{dt} = A$

$A + t(At+B) = 2t$

$At^2 + Bt + A = 2t$

$B = 2$ ✓
$A = 0$

$x_p(t) = 2$

③ $x(t) = x_h(t) + x_p(t)$

$x(t) = Ae^{-\frac{1}{2}t^2} + 2$

Q: What if guessing does not work?

A: $\boxed{\text{VARIATION OF PARAMETERS}}$ [VOP]

SITUATION: $\frac{dx}{dt} + p(t)\,x = q(t)$

① FIND HOMOG SOLNS $x_h(t) = C\,x_1(t)$

$\text{Span}\{x_1(t)\} \Leftrightarrow$ HOMOG SOLNS.

Seek PART. SOLN $x(t) = v(t)\,x_1(t) = v x_1$

PLUG INTO ODE: $\frac{dx}{dt} = v\frac{dx_1}{dt} + \frac{dv}{dt}x_1$

$$v\frac{dx_1}{dt} + \frac{dv}{dt}x_1 + p v x_1 = q$$

$$v\Big[\underbrace{\frac{dx_1}{dt} + p x_1}_{=0 \text{ because } x_1 \text{ homog soln.}}\Big] + x_1\frac{dv}{dt} = q$$

$$x_1 \frac{dv}{dt} = g \Rightarrow \frac{dv}{dt} = \frac{g(t)}{x_1(t)}$$

$$\boxed{v(t) = \int \frac{g(t)}{x_1(t)} dt} \Rightarrow x_p = v(t)\, x_1(t)$$

Ex: $\frac{dx}{dt} + tx = 2t = g(t)$

$$x_h(t) = A e^{-\frac{1}{2}t^2} \quad \text{(earlier)}$$

$$x_1(t) = e^{-\frac{1}{2}t^2}$$

$$v(t) = \int \frac{2t}{e^{-\frac{1}{2}t^2}} dt = 2\int e^{\frac{1}{2}t^2} \underbrace{t\,dt}_{du}$$

e^u

$$v(t) = 2e^{\frac{1}{2}t^2}$$

$$x_p(t) = 2e^{\frac{1}{2}t^2} e^{-\frac{1}{2}t^2} = 2$$

Ex: $\frac{dx}{dt} + \frac{5}{t}x = 7t$

Homog: $\frac{dx}{dt} + \frac{5}{t}x = 0 \quad \frac{dx}{dt} = -\frac{5x}{t}$

$$\int \frac{dx}{x} = -\int \frac{5\,dt}{t} \qquad e^{\ln|x|} = e^{-5\ln t + c}$$

$$x_h(t) = A\, t^{-5} = \frac{A}{t^5}$$

$$x_1(t) = t^{-5} = \frac{1}{t^5}$$

PART $x_1(t) = \frac{1}{t^5}$ $f(t) = 7t$

$$v(t) = \int \frac{f(t)}{v_1(t)} dt = \int \frac{7t}{t^{-5}} dt$$

$$= \int 7t^6 dt = t^7$$

$$x_p(t) = v(t)\, x_1(t) = t^7 \cdot \frac{1}{t^5} = t^2$$

$\Rightarrow$ General $\boxed{x(t) = \frac{A}{t^5} + t^2}$

If $x(1) = 4 \Rightarrow 4 = \frac{A}{1} + 1 = A + 1$ $A = 3$

$$x(t) = \frac{3}{t^5} + t^2$$

Problem: Solve $\ddot{x} + 3\dot{x} + 2x = 5\sin t$

$x(0) = 1,\ \dot{x}(0) = 2$

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = 5\sin t$$

HOMOG: Seek exponential solutions $x = e^{rt}$

$$(\underbrace{r^2 + 3r + 2}_{p(r)})e^{rt} = 0$$

$p(r)$ characteristic polynomial.

$p(r) = r^2 + 3r + 2 = (r+2)(r+1) = 0$

$r = -2$ OR $r = -1$ $\{e^{-t}, e^{-2t}\}$

$$x_h(t) = C_1 e^{-t} + C_2 e^{-2t}$$

Q: Are these all homog solns?

$\ddot{x} + 3\dot{x} + 2x = 0$ HOMOG.

OPERATORS $D = \frac{d}{dt}$

$(D^2 + 3D + 2I)\, x(t) = 0$

$(D + 2I) \circ \Big[\underbrace{(D + I)\, x(t)}_{u(t)}\Big] = 0$

$(D + 2I)\, u(t) = 0 \qquad \frac{du}{dt} + 2u = 0$

$\frac{du}{dt} = -2u \;\rightarrow\; u(t) = c_1 e^{-2t}$

$\frac{dx}{dt} + x = c_1 e^{-2t}$ Integ facto e^t

$\underbrace{e^t \frac{dx}{dt} + e^t x}_{\frac{d}{dt}[e^t x]} = c_1 e^{-t}$

$\Rightarrow e^t x(t) = -c_1 e^{-t} + C_2$

$\Rightarrow x(t) = -c_1 \underline{\underline{e^{-2t}}} + C_2 \underline{\underline{e^{-t}}}$

$\mathrm{Span}\{e^{-2t}, e^{-t}\}$

PART SOLUTION $\ddot{x} + 3\dot{x} + 2x = 5\sin t$

(2) $x_p = a\cos t + b\sin t$

(3) $\dot{x}_p = b\cos t - a\sin t$

(1) $\ddot{x}_p = -a\cos t - b\sin t$

$(a + 3b)\cos t + (-3a + b)\sin t = 5\sin t$

$$a+3b=0 \qquad a=-3b \qquad b=1/2$$
$$-3a+b=5 \qquad 10b=5 \qquad a=-3/2$$

$$x_p(t) = -\frac{3}{2}\cos t + \frac{1}{2}\sin t$$
$$= \frac{1}{2}(-3\cos t + \sin t)$$

$$x(t) = \overbrace{c_1 e^{-2t} + c_2 e^{-t}}^{\text{transient}} \cancel{+} -\frac{3}{2}\cos t + \frac{1}{2}\sin t$$

$$\dot{x}(t) = -2c_1 e^{-2t} - c_2 e^{-t} + \frac{3}{2}\sin t + \frac{1}{2}\cos t$$

$$x(0) = c_1 + c_2 - \frac{3}{2} = 1 \qquad c_1 + c_2 = 5/2$$
$$\dot{x}(0) = -2c_1 - c_2 + \frac{1}{2} = 2 \qquad -2c_1 - c_2 = 3/2$$
$$-c_1 = 4$$
$$c_2 = \frac{5}{2} + 4 = \frac{13}{2} \qquad c_1 = -4$$

$$x(t) = -4e^{-2t} + \frac{13}{2}e^{-t} \underbrace{-\frac{3}{2}\cos t + \frac{1}{2}\sin t}_{\text{Rectangular Form}} \checkmark$$

Amplitude-phase FORM

$\Downarrow$

$$a\cos\omega t + b\sin\omega t = A\cos(\omega t - \phi)$$

A amplitude

ϕ phase angle

Sum of angles formula for cosine:

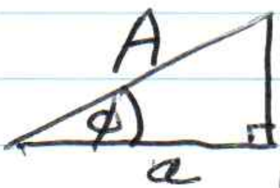
$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$

$$A\cos(\omega t-\phi) = A\cos\omega t\cos\phi + A\sin\omega t\sin\phi$$

$$= (A\cos\phi)\cos\omega t + (A\sin\phi)\sin\omega t$$

$$= a\cos\omega t + b\sin\omega t$$

$$\Rightarrow A\cos\phi = a \qquad A\sin\phi = b$$



$A = \sqrt{a^2+b^2}$

$\tan\phi = b/a$ $\quad$ * depends on quadrant

Best to use picture to remember

$$x_p(t) = -\tfrac{3}{2}\cos t + \tfrac{1}{2}\sin t$$

$$= -\tfrac{1}{2}\left[3\cos t - \sin t\right]$$

$a = 3$

$b = -1$

$A = \sqrt{9+1}$

$= \sqrt{10}$

$$-\tfrac{1}{2}\sqrt{10}\cos(t-\phi)$$

Input Ampl. $= 5$

Resp. Ampl. $= \frac{\sqrt{10}}{2}$

$\phi = \tan^{-1}(-1/3)$