Reference: Lecture #6 Notes

Solve $\ddot{x} + 3\dot{x} + 2x = t^2 e^{3t}$

LTI ODE (LINEAR TIME INVARIANT)

UNDET COEFFS.
$x(t) = (at^2 + bt + c)e^{3t}$ works

Exponential Shift Rule:

SITUATION $[p(D)]\, x(t) = e^{rt} f(t)$ (*)

$[p(D) = D^2 + 3D + 2I$ in example$]$

Solution: Solve $[p(D + rI)]\, u(t) = f(t) \Rightarrow u(t)$

Then $x(t) = e^{rt} u(t)$ solves (*)

Proof: $D = \frac{d}{dt}$

$$D(e^{rt}u) = e^{rt} Du + r e^{rt} u$$
$$= e^{rt}(Du + ru)$$
$$= e^{rt}[D + rI]\, u(t)$$

$$D^2(e^{rt} u(t)) = D[D(e^{rt}u)]$$
$$= D[e^{rt}(D + rI)\, u(t)]$$
$$= e^{rt}(D^2 + rD)\, u(t) + r e^{rt}(D + rI)\, u(t)$$
$$= e^{rt}[D^2 + 2rD + r^2 I]\, u(t)$$
$$= e^{rt}(D + rI)^2\, u(t)$$

etc.

$$D^n(e^{rt} u(t)) = e^{rt}(D + rI)^n\, u(t)$$

Apply to $x(t) = e^{rt} u(t)$ in ⊛

$$[p(D)]\left(e^{rt} u(t)\right) = e^{rt} p(D + rI)\, u(t) = e^{rt} g(t)$$

$$\therefore\ [p(D + rI)]\, u(t) = g(t) \qquad \ddagger$$

Solve this to get $u(t)$.

$\Rightarrow$ $x(t) = e^{rt} u(t)$ solves ⊛

Ex: Solve $\ddot{x} + 3\dot{x} + 2x = t^2 e^{3t}$

$r = 3 \quad g(t) = t^2$

Focus on characteristic polynomial.

$$p(s) = s^2 + 3s + 2 = (s+2)(s+1)$$

$$p(s+3) = (s+5)(s+4) = s^2 + 9s + 20$$

$$p(D + rI) = D^2 + 9D + 20I$$

Solve $\ddot{u} + 9\dot{u} + 20u = t^2$

$$\begin{aligned}
20 \quad & u(t) = at^2 + bt + c \\
9 \quad & \dot{u} = 2at + b \\
1 \quad & \ddot{u} = 2a
\end{aligned}$$

$$\ddot{u} + 9\dot{u} + 20u = 20at^2 + (20b + 18a)t + (20c + 9b + 2a) = t^2$$

$20a = 1 \longrightarrow a = 1/20$

$18a + 20b = 0 \qquad b = -9/200$

$2a + 9b + 20c = 0 \qquad c = \frac{61}{4000}$

$$u(t) = \frac{1}{20}t^2 - \frac{9}{200}t + \frac{61}{4000}$$

Soln: $x(t) = e^{3t}\left(\frac{1}{20}t^2 - \frac{9}{200}t + \frac{61}{4000}\right)$

Solve $\ddot{x} + 3\dot{x} + 2x = e^{-t} \cdot 1$ $r=-1$

$$p(s) = s^2 + 3s + 2 = (s+2)(s+1) = 0$$

$$s = -2, \; s = -1$$

ERF FAILS

RRF $\quad x_p(t) = \dfrac{te^{-t}}{p'(-1)} = te^{-t}$ ✓

$p'(s) = 2s+3 \quad p'(-1) = -2+3 = 1$

$r=-1 \quad p(s-1) = (s+1)s = s^2 + s$

<u>ESR</u> Solve $\ddot{u} + \dot{u} = 1$

undet coeff, or observe $u(t) = t$

$\therefore \; x(t) = u(t) \cdot e^{rt} = te^{-t}$ ✓

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method of LAST RESORT

<u>VARIATION OF PARAMETERS</u>

Already done for $\dfrac{dx}{dt} + p(t)x = g(t)$ 1st order

Solve homog $\Rightarrow x(t) = c_1 x_1(t)$

"VARY THE PARAM" seek $x_p(t) = v_1(t) x_1(t)$

$$\Rightarrow \frac{dv_1}{dt} = \frac{g(t)}{x_1(t)} \Rightarrow v_1(t).$$

$$\Rightarrow x_p(t) = v_1(t) x_1(t)$$

Higher order versions

Solve $\ddot{x} + p_1(t)\dot{x} + p_0(t)x = R(t)$

$$\ddot{x} + p_1\dot{x} + p_0 x = R$$

FIRST solve HOMOG ODE $\Rightarrow x_h(t) = \underline{c_1} x_1(t) + \underline{c_2} x_2(t)$

$$= \text{span}\{x_1(t), x_2(t)\}$$

"VARY THE PARAMETERS"

seek $\boxed{x_p(t) = v_1 x_1 + v_2 x_2}$ $\quad v_1 = v_1(t)$, $v_2 = v_2(t)$

$$\dot{x} = v_1\dot{x}_1 + \dot{v}_1 x_1 + v_2\dot{x}_2 + \dot{v}_2 x_2$$

$$\boxed{\dot{x} = (v_1\dot{x}_1 + v_2\dot{x}_2)} + \cancel{(x_1\dot{v}_1 + x_2\dot{v}_2)}$$

Set $\boxed{x_1\dot{v}_1 + x_2\dot{v}_2 = 0}$ Assume this!

$$\boxed{\ddot{x} = v_1\ddot{x}_1 + \dot{v}_1\dot{x}_1 + v_2\ddot{x}_2 + \dot{v}_2\dot{x}_2}$$

So $\ddot{x} + p_1\dot{x} + p_0 x$

$$= \underline{v_1\ddot{x}_1} + \dot{v}_1\dot{x}_1 + \underline{v_2\ddot{x}_2} + \dot{v}_2\dot{x}_2$$

$$+ p_1(\underline{v_1\dot{x}_1} + v_2\dot{x}_2) + p_0(\underline{v_1 x_1} + \underline{v_2 x_2})$$

$$= v_1\cancel{(\ddot{x}_1 + p_1\dot{x}_1 + p_0 x_1)}^{0} + v_2\cancel{(\ddot{x}_2 + p_1\dot{x}_2 + p_0 x_2)}^{0}$$

$$+ \dot{x}_1\dot{v}_1 + \dot{x}_2\dot{v}_2$$

$$= \boxed{\dot{x}_1\dot{v}_1 + \dot{x}_2\dot{v}_2 = R}$$

$$\left\{\begin{matrix} x_1\dot{v}_1 + x_2\dot{v}_2 = 0 \\ \dot{x}_1\dot{v}_1 + \dot{x}_2\dot{v}_2 = R \end{matrix}\right\} \Rightarrow \begin{bmatrix} x_1 & x_2 \\ \dot{x}_1 & \dot{x}_2 \end{bmatrix}\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ R \end{bmatrix}$$

WRONSKIAN MATRIX $W(t)$

$$\left[W(t)\right] \begin{bmatrix} \dot v_1 \\ \dot v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ R \end{bmatrix}$$

2×2 matrix

$$\Rightarrow \begin{bmatrix} \dot v_1 \\ \dot v_2 \end{bmatrix} = \left[W(t)\right]^{-1} \begin{bmatrix} 0 \\ R \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\det = ad - bc$$

$$\begin{bmatrix} \dot v_1 \\ \dot v_2 \end{bmatrix} = \frac{1}{\underbrace{x_1 \dot x_2 - x_2 \dot x_1}_{\det[W(t)]}} \begin{bmatrix} \dot x_2 & -x_2 \\ -\dot x_1 & x_1 \end{bmatrix} \begin{bmatrix} 0 \\ R \end{bmatrix}$$

$$= \frac{1}{x_1 \dot x_2 - x_2 \dot x_1} \begin{bmatrix} -x_2 R \\ x_1 R \end{bmatrix}$$

$$\boxed{\begin{aligned} \frac{dv_1}{dt} &= \frac{-x_2 R}{x_1 \dot x_2 - x_2 \dot x_1} \\ \frac{dv_2}{dt} &= \frac{x_1 R}{x_1 \dot x_2 - x_2 \dot x_1} \end{aligned}}$$

Solve $\Rightarrow v_1(t), v_2(t)$

$$x_p(t) = v_1(t) x_1(t) + v_2(t) x_2(t)$$

Generalizes to higher order.

3rd order

$$x_h(t) = c_1 x_1(t) + c_2 x_2(t) + c_3 x_3(t)$$

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ \dot x_1 & \dot x_2 & \dot x_3 \\ \ddot x_1 & \ddot x_2 & \ddot x_3 \end{bmatrix} \begin{bmatrix} \dot v_1 \\ \dot v_2 \\ \dot v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ R \end{bmatrix} \Rightarrow \begin{matrix} v_1(t) \\ v_2(t) \\ v_3(t) \end{matrix}$$

$$\Rightarrow x_p(t) = v_1 x_1 + v_2 x_2 + v_3 x_3$$

and so on