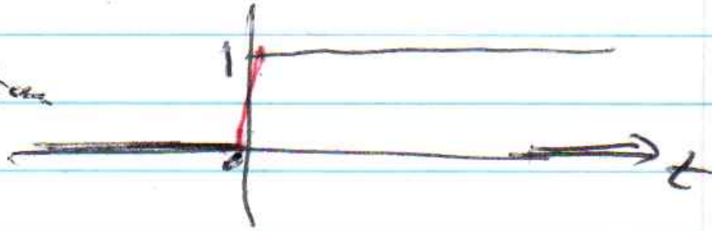
RECAP

Heaviside function

$u(t)$



DERIVATIVE of $u(t)$

DELTA FUNCTION $\delta(t) = u'(t) = \begin{cases} 0 & t<0 \\ \infty & t=0 \\ 0 & t>0 \end{cases}$

Generalized Derivative

$$\int_{-\infty}^{+\infty} \delta(t)\,dt = 1$$

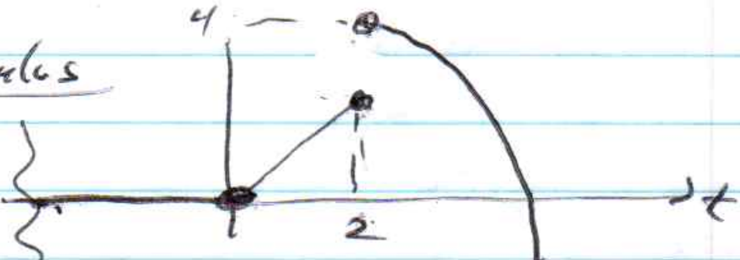
[not Really a function. A "distribution" or "probability density".]

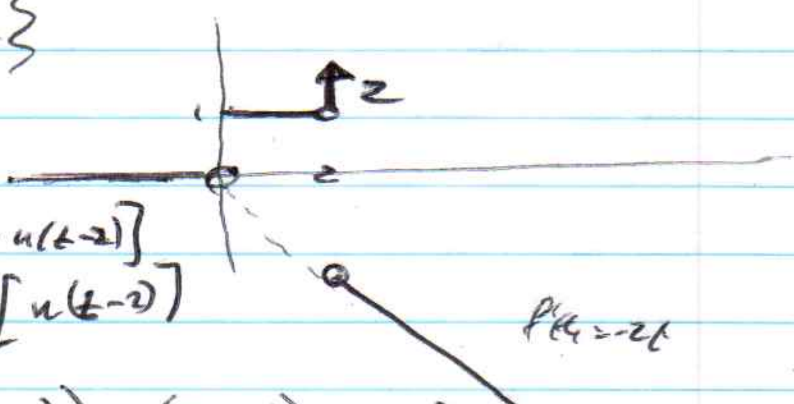
Translated Heaviside $u(t-a)$

translated Delta function. $\delta(t-a) = u'(t-a)$

Generalized Calculus

$$f(t) = \begin{cases} 0 & t \le 0 \\ t & 0 < t \le 2 \\ 8-t^2 & t>2 \end{cases}$$





$$f(t) = 0 + t\left[u(t) - u(t-2)\right] + (8-t^2)\left[u(t-2)\right]$$

$$f(t) = t\left(u(t) - u(t-2)\right) + (8-t^2)\,u(t-2)$$

$$f'(t) = t\left[\delta(t) - \delta(t-2)\right] + \left[u(t) - u(t-2)\right] + (8-t^2)\,\delta(t-2) - 2t\,u(t-2)$$

$$= 0 - 2\delta(t-2) + 4\delta(t-2) + u(t) - u(t-2) - 2t\,u(t-2)$$

$$f(a) = \int_{-\infty}^{+\infty} f(t)\,\delta(t-a)\,dt$$

$$f(0) = \int_{-\infty}^{+\infty} f(t)\,\delta(t)\,dt$$

→ $= 2\delta(t-2) + 1[u(t) - u(t-2)] - 2t\,u(t-2)$

THE BIG IDEA : LAPLACE TRANSFORM

$$\mathcal{L}[f(t)] = F(s) = \int_{0^-}^{+\infty} e^{-st} f(t)\,dt$$

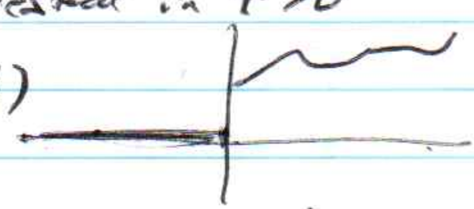
$$\mathcal{L}[x(t)] = X(s) = \int_{0^-}^{+\infty} e^{-st} x(t)\,dt$$

$\mathcal{L}$ linear

$$\mathcal{L}[1] = \mathcal{L}[u(t)] = \frac{1}{s}$$

Note: only interested in $t>0$

$f(t)\,u(t)$



$$\mathcal{L}[t] = \mathcal{L}[t\,u(t)] = \frac{1}{s^2}$$

$$\mathcal{L}[t^2] = \mathcal{L}[t^2 u(t)] = \frac{2}{s^3}$$

$\vdots$

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$$

s - Derivative Rule

$$\mathcal{L}[t\,f(t)] = -F'(s)$$

why? $$\text{"} \int_{0^-}^{+\infty} e^{-st}\, t\,f(t)\,dt = \cdots$$

$$F(s) = \mathcal{L}[f(t)] = \int_{0^-}^{+\infty} e^{-st} f(t)\,dt$$

$$F'(s) = -\int_{0^-}^{+\infty} e^{-st}[t\,f(t)]\,dt$$

$$= -\mathcal{L}[t\,f(t)]$$

$$\mathcal{L}[t\,f(t)] = -F'(s)$$

ex: $\mathcal{L}[t] = \frac{1}{s^2}$

$$\mathcal{L}[t^2] = \mathcal{L}[t\cdot t] = +\frac{2}{s^3}$$

$$\mathcal{L}[t^3] = \mathcal{L}[t\cdot t^2] = +\frac{3!}{s^4}$$

etc.

$$\mathcal{L}[e^{at}] = \int_{0^-}^{+\infty} e^{-st} e^{at}\,dt$$

$$= \int_{0^-}^{+\infty} e^{-(s-a)t}\,dt = \cdots = \frac{1}{s-a}$$

$$\mathcal{L}[t e^{at}] = +\frac{1}{(s-a)^2}$$

s-shift Rule : $\mathcal{L}\left[e^{rt}f(t)\right] = F(s-r)$

Why?

$$\mathcal{L}\left[e^{rt}f(t)\right] = \int_{0^-}^{+\infty} e^{-st}e^{rt}f(t)\,dt$$

$$= \int_{0^-}^{+\infty} e^{-(s-r)t}f(t)\,dt = F(s-r)$$

ex: $\mathcal{L}\left[te^{at}\right] = \frac{1}{(s-a)^2}$

$\mathcal{L}[t] = \frac{1}{s^2}$

$$\mathcal{L}[\delta(t)] = \int_{0^-}^{+\infty} e^{-st}\delta(t)\,dt = 1$$

$$e^{i\omega t} = \cos\omega t + i\sin\omega t$$

$$\mathcal{L}[\cos\omega t] = \frac{s}{s^2+\omega^2}$$

$$\mathcal{L}[\sin\omega t] = \frac{\omega}{s^2+\omega^2}$$

Inverse LAPLACE TRANSFORM $\mathcal{L}^{-1}$

$$F(s) = \frac{3}{s^2} + \frac{4s}{s^2+9} + \frac{2}{s^2+9} \quad \omega=3$$

Recognition + LAPLACE TRANSFORM

$$f(t) = 3t + 4\cos 3t + \frac{2}{3}\sin 3t$$

TRANSFORM DERIVATIVES

$$f(t) \qquad \mathcal{L}[f(t)] = F(s)$$

$$\mathcal{L}[f'(t)] = \int_{0^-}^{+\infty} e^{-st} f'(t)\,dt$$

$$u = e^{-st} \qquad dv = f'(t)\,dt$$

$$du = -se^{-st}\,dt \qquad v = f(t)$$

$$= e^{-st} f(t)\Big|_{0^-}^{+\infty} + s\boxed{\int_{0^-}^{+\infty} e^{-st} f(t)\,dt}$$

$$= 0 - f(0) + s\,F(s)$$

$$\mathcal{L}[f'(t)] = sF(s) - f(0)$$

$$\left[\mathcal{L}[\dot{x}(t)] = s\,X(s) - x(0)\right]$$

$$f''(t) = \frac{d}{dt}[f'(t)] \qquad f'''(t) = \frac{d}{dt}[f''(t)]$$

$$\mathcal{L}[f''(t)] = s[\mathcal{L}(f'(t))] - f'(0)$$

$$= s[sF(s) - f(0)] - f'(0)$$

$$= s^2 F(s) - s f(0) - f'(0)$$

$$\mathcal{L}[f'''(t)] = s\mathcal{L}[f''(t)] - f''(0)$$

$$= s^3 F(s) - s^2 f(0) - s f'(0) - f''(0)$$

etc,

$$\left[\mathcal{L}[\dddot{x}(t)] = s^3 X(t) - s^2 x(0) - s\dot{x}(0) - \ddot{x}(0)\right]$$

SPECIAL CASE : REST INITIAL CONDITIONS.

$$x(0)=0,\quad \dot{x}(0)=0,\quad \ddot{x}(0)=0,\ \text{etc.}$$

SIMPLER:

$$\mathcal{L}[\dot{x}(t)] = s\,X(s)$$

$$\mathcal{L}[\ddot{x}(t)] = s^2\,X(s)$$

$$\vdots$$

$$\mathcal{L}[x^{(n)}(t)] = s^n\,X(s)$$

$$[p(D)]\,x(t) \qquad [= \ddot{x} + 3\dot{x} + 2x]$$

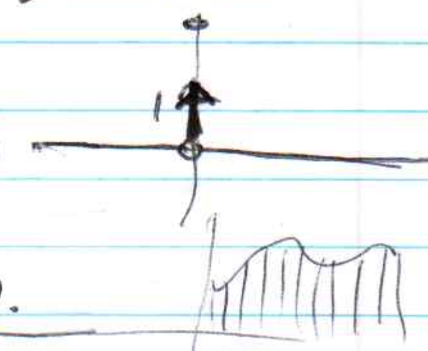
$$\mathcal{L}[(p(D))\,x(t)] = p(s)\,X(s)$$

Apply this to solution of ODE'S

UNIT IMPULSE RESPONSE

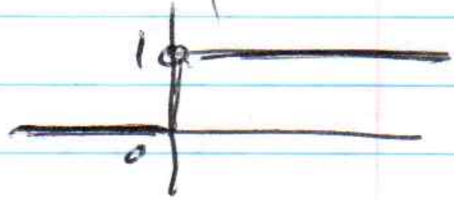
Solve $[p(D)]\,x(t) = \delta(t)$

solution is also called the weight function $w(t)$.



UNIT STEP RESPONSE

$$[p(D)]\,x(t) = u(t)$$



In both, assume REST INITIAL CONDITIONS

EXAMPLES:

$P(D) = D + 3I$ unit impulse and unit step responses.

① Impulse $\boxed{\dot{x} + 3x = \delta(t)}$ $x(0)=0$

② STEP $\dot{x} + 3x = u(t)$ $x(0)=0$

① TRANSFORM BOTH SIDES

$$\mathcal{L}[x(t)] = X(s)$$

$$(s+3)\,X(s) = 1$$

$$X(s) = \frac{1}{s+3} = \frac{1}{p(s)}$$

$$\Rightarrow \quad x(t) = \boxed{e^{-3t} = w(t)}$$

Remind: $\mathcal{L}(e^{at}) = \dfrac{1}{s-a}$

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② $\dot{x} + 3x = u(t) = \begin{cases} 0 & t<0 \\ 1 & t>0 \end{cases}$

$$\Big\downarrow \mathcal{L}$$

$$(s+3)X(s) = \frac{1}{s} \qquad \boxed{X(s) = \frac{1}{s(s+3)}}$$

$$X(s) = \frac{1}{s(s+3)} = \frac{A}{s} + \frac{B}{s+3}$$

$$1 = A(s+3) + Bs = (A+B)s + 3A$$

$$A+B=0 \quad 3A=1 \rightarrow A = 1/3 \quad B = -\tfrac{1}{3}$$

$$X(s) = \frac{1}{3}\left(\frac{1}{s} - \frac{1}{s+3}\right)$$

$$x(t) = \frac{1}{3}\left[1 - e^{-3t}\right]$$

CHECK: $\boxed{\dot{x}(t) = +e^{-3t}}$

EXAMPLES HARMONIC RESPONSE

$$\ddot{x} + \omega^2 x = \delta(t)$$ Rest Initial Condition.

Unit Impulse Response for $D^2 + \omega^2 I$

TRANSFORM

$$\ddot{x} + \omega^2 x = \delta(t) \qquad p(s) = s^2 + \omega^2$$

$$\downarrow \mathcal{L}$$

$$(s^2 + \omega^2)\, X(s) = 1 \qquad x(0) = 0$$

$$X(s) = \frac{1}{s^2 + \omega^2} = \frac{1}{p(s)} \qquad \dot{x}(0) = 0$$

$$X(s) = \frac{1}{\omega}\left(\frac{\omega}{s^2 + \omega^2}\right)$$

$$\therefore \quad \boxed{x(t) = \frac{1}{\omega} \sin \omega t}$$

LAPLACE DIRECT

Solve $\dot{x} + 3x = 3\cos 2t \qquad x(0) = 2$

$$\downarrow \mathcal{L}$$

$$(sX - 2) + 3X = 3\,\frac{s}{s^2 + 4}$$

$$(s+3)X - 2 = \frac{3s}{s^2 + 4}$$

$$(s+3)X = 2 + \frac{3s}{s^2+4} = \frac{2s^2 + 3s + 8}{s^2 + 4}$$

$$X(s) = \frac{2s^2+3s+8}{(s+3)(s^2+4)} \qquad \text{Find } x(t)$$

PARTIAL FRACTIONS

$$\frac{2s^2+3s+8}{(s+3)(s^2+4)} = \frac{A}{s+3} + \frac{Bs+C}{s^2+4} \qquad A, B, C\ ?$$

$$\boxed{2s^2+3s+8 = A(s^2+4) + (Bs+C)(s+3)}$$

$$= (A+B)s^2 + (3B+C)s + (4A+3C)$$

$$\left\{\begin{array}{l} A+B = 2 \\ 3B+C = 3 \\ 4A + 3C = 8 \end{array}\right\} \rightarrow A, B, C.$$

Pick particular points.

$$s=-3 \qquad 2(9)+3(-3)+8 = 17 = 13A \qquad \boxed{A = \frac{17}{13}}$$

use complex numbers.

$$s = 2i \qquad -8+8+6i = (B\cdot 2i + C)(3+2i)$$

$$6i = 6Bi + 3C - 4B + 2Ci$$

$$6i = (-4B+3C) + (6B+2C)i$$

$$\left\{\begin{array}{l} -4B+3C=0 \\ 6B+2C=6 \end{array}\right\} \rightarrow \boxed{B = 9/13} \ \boxed{C = 12/13}$$

$$X(s) = \frac{17}{13}\left(\frac{1}{s+3}\right) + \frac{9}{13}\left(\frac{s}{s^2+4}\right) + \frac{\cancel{12}\ 6}{13}\left(\frac{\cancel{1}\ 2}{s^2+4}\right)$$

$$x(t) = \frac{17}{13} e^{-3t} + \frac{9}{13}\cos 2t + \frac{6}{13}\sin 2t$$

Solve $\ddot{x} + 3\dot{x} + 2x = 4$ $\quad x(0) = 0 \quad \dot{x}(0) = 0$

Rest Initial Condition.

Zero state Response (ZSR)
Zero Input Response (ZIR)

ZIR + ZSR

$$(s^2 + 3s + 2) X(s) = \frac{4}{s}$$

$(s+2)(s+1)$

$$X(s) = \frac{4}{s(s+1)(s+2)}$$

$$\frac{4}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

$$4 = A(s+1)(s+2) + Bs(s+2) + Cs(s+1)$$

$s=0 \quad 4 = 2A \quad A = 2$

$s=-1 \quad 4 = -B \quad B = -4$

$s=-2 \quad 4 = 2C \quad C = 2$

$$X(s) = \frac{2}{s} - \frac{4}{s+1} + \frac{2}{s+2}$$

$$\boxed{x(t) = 2 - 4e^{-t} + 2e^{-2t}}$$