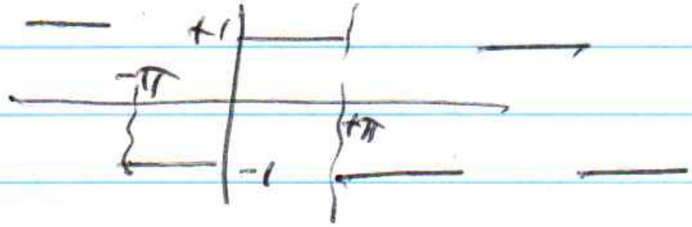
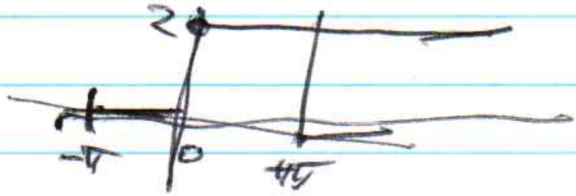
LAST DETAILS OF FOURIER SERIES

$Sq(t)$



$1 + Sq(t)$



$$Sq(t) = \frac{4}{\pi} \sum_{n\ odd} \frac{\sin nt}{n}$$

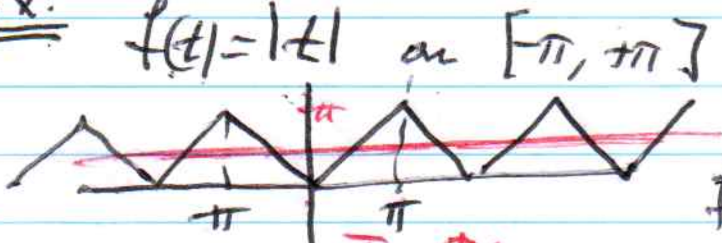
$$\frac{1}{2}(1 + Sq(t)) =$$

neither odd nor even.

$$\frac{1}{2}(1 + Sq(t)) = \frac{1}{2}\left(1 + \frac{4}{\pi} \sum_{n\ odd} \frac{\sin nt}{n}\right)$$

$$\frac{1}{2} + \frac{2}{\pi} \sum_{n\ odd} \frac{\sin nt}{n}$$

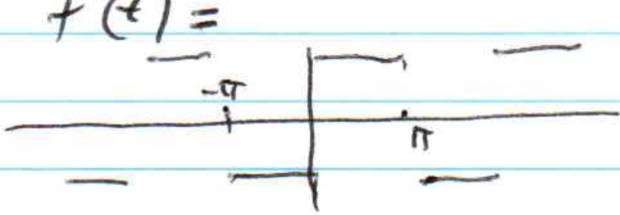
Ex: $f(t) = |t|$ on $[-\pi, +\pi]$ even



$\bar{f} = \pi/2$

$$f(t) = \begin{cases} -t & [-\pi, 0) \\ t & [0, \pi) \end{cases}$$

$f'(t) =$



$$f'(t) = \begin{cases} -1 & [-\pi, 0) \\ +1 & [0, \pi) \end{cases}$$

$= Sq(t)$

$$f'(t) \approx \frac{4}{\pi}\left[\sum_{n\,odd} \frac{\sin nt}{n}\right]$$

$$f(t) = \underset{\substack{\| \\ \pi/2 = \bar{f}}}{C} + \frac{4}{\pi}\sum_{n\,odd}\left[\frac{-\cos nt}{n^2}\right]$$

$\left[\text{Can also compute } C = \frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt\right]$

$$|t| = f(t) \sim \frac{\pi}{2} - \frac{4}{\pi}\sum_{n\,odd}\frac{\cos nt}{n^2}$$

FS on the cheap.

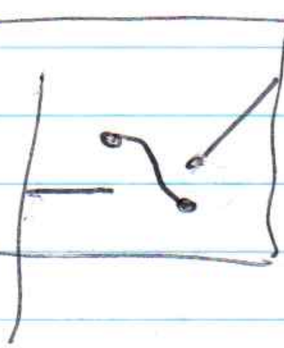
<u>DIRECTLY</u>: $f(t) \cong \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt)$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}|t|\,dt \qquad \text{all } b_n = 0$$

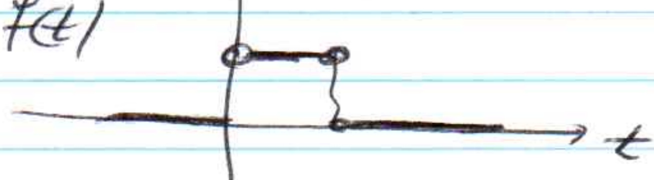
$$a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi}|t|\cos nt\,dt$$

DISCONTINUOUS INPUTS

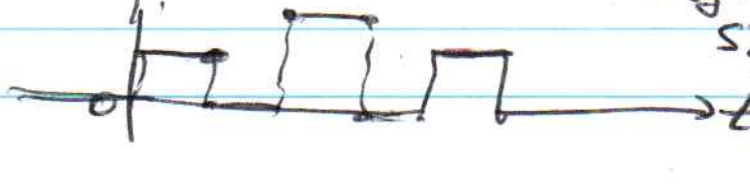
$[P(D)]\,x(t) = f(t)$



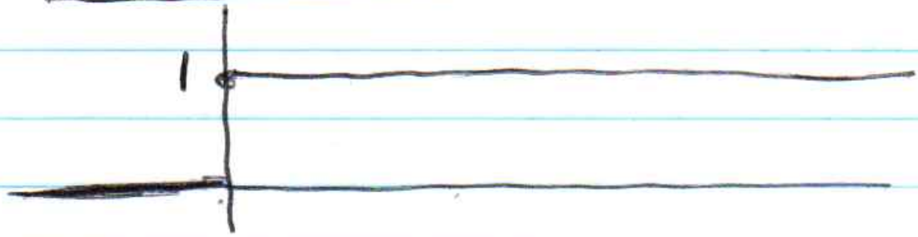
$f(t)$



Digital Signal

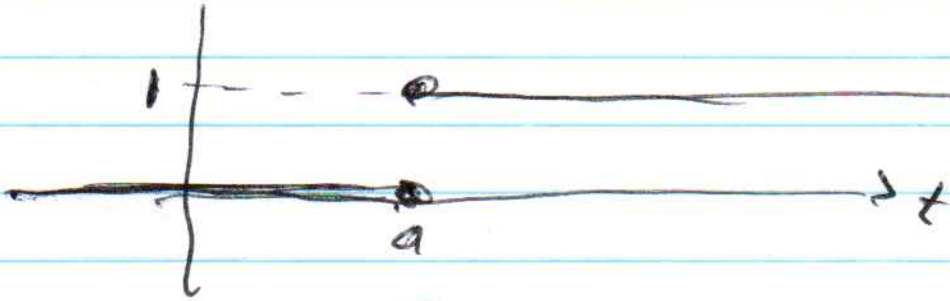


Heaviside function



$$u(t) = \begin{cases} 0 & t<0 \\ 1 & t \geq 0 \end{cases}$$

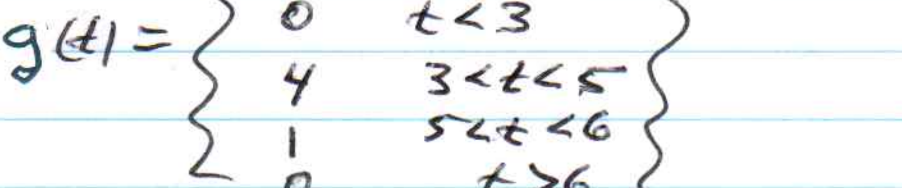
translated Heaviside function.



$$u_a(t) = \begin{cases} 0 & t<a \\ 1 & t>a \end{cases} = u(t-a)$$

Box Functions

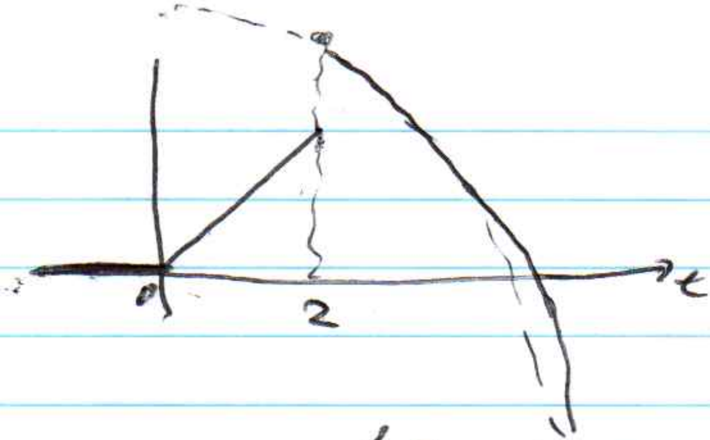
EXAMPLE



$$g(t) = \begin{cases} 0 & t<3 \\ 4 & 3<t<5 \\ 1 & 5<t<6 \\ 0 & t>6 \end{cases}$$

$$= 4\left[u_3(t) - u_5(t)\right] + 1\left[u_5(t) - u_6(t)\right]$$

$$= 4\, u_3(t) - 3\, u_5(t) - u_6(t)$$



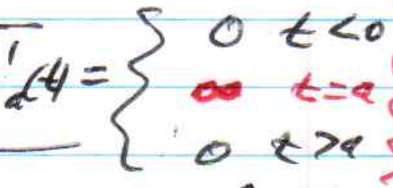
$$f(t) = \begin{cases} 0 & t<0 \\ t & 0<t<2 \\ 4-t^2 & t>2 \end{cases}$$

$$f(t) = t\left[u(t) - u_2(t)\right] + (4-t^2)\,u_2(t)$$

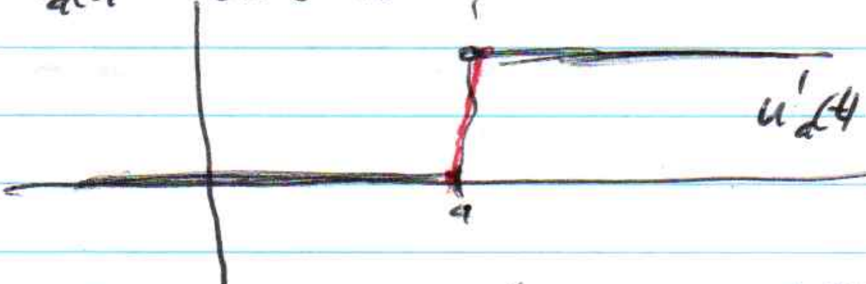
DERIVATIVE OF HEAVISIDE FUNCTION

$$u(t) = \begin{cases} 0 & t<0 \\ 1 & t>0 \end{cases}$$

$$u'(t) = \begin{cases} 0 & t<0 \\ \infty & t=0 \\ 0 & t>0 \end{cases} = \delta(t)$$



$u_a(t) = u(t-a)$



$$u_a'(t) = \begin{cases} 0 & t<0 \\ \infty & t=a \\ 0 & t>a \end{cases}$$

$= \delta_a(t)$

$= \delta(t-a)$

Can we make better sense of this?

BIG IDEA : DELTA FUNCTION

MAKE SENSE OF FOLLOWING :

For any function $f(t)$

$$f(0) = \int_{-\infty}^{+\infty} f(t)\, \underline{\delta(t)}\, dt$$

$$f(a) = \int_{-\infty}^{+\infty} f(t)\, \delta_{\underline{a}}(t)\, dt$$

* EVALUATION of a function is same as "integrating against a delta function".

LINEAR FUNCTIONALS

* $\mathbb{R}^n$ $\qquad \overrightarrow{\text{vector}} \longrightarrow$ number

‡ "functions" $\qquad \boxed{\text{function}} \longrightarrow$ number.

$\vec{v} = \langle v_1, v_2, v_3 \rangle =$

$= v_1 \vec{i} + v_j \vec{j} + v_3 \vec{k}$

$\text{comp}_x(\vec{V}) = v_1$

$\text{comp}_y(\vec{v}) = v_2$

$\text{comp}_z(\vec{v}) = v_3$

component functions are Linear functionals.

‡ Fourier coefficients. $f(t) = sq(t)$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(t)\,dt \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(t)\cos nt\,dt$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(t)\sin nt\,dt$$

these are also linear functionals

Key Idea: Generalized Derivative.

[BASED on Integration by Parts]

Product Rule: $\frac{d}{dt}[uv] = u\frac{dv}{dt} + v\frac{du}{dt}$

Integrate from $t=a$ to $t=b$ using FTC

$$\int_a^b \frac{d}{dt}[uv]\,dt = \int_a^b \left(u\frac{dv}{dt}\right)dt + \int_a^b \left(v\frac{du}{dt}\right)dt$$

$$\left\{\begin{array}{l}\left[\ uv = \int u\,dv - \int v\,du\ \right] \\ \left[\ \int u\,dv = uv - \int v\,du\ \right]\end{array}\right\}$$

$$\to \left[u(t)\,v(t)\right]\Big|_{t=a}^{t=b} = \int_a^b u(t)\,v'(t)\,dt + \int_a^b v(t)\,u'(t)\,dt$$

Improper Integrals. $\underline{[a,b] \to (-\infty,+\infty)}$

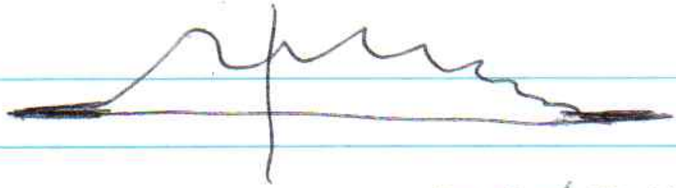
$$\int_a^b u(t)\,v'(t)\,dt = \left[u(t)\,v(t)\right]\Big|_{t=a}^{t=b} - \int_a^b v(t)\,u'(t)\,dt$$

$u(t) = g(t) \qquad v(t) = f(t)$

$$\int_a^b f'(t)\,g(t)\,dt = \cancel{\left[f(t)\,g(t)\right]_a^b} - \int_a^b f(t)\,g'(t)\,dt$$

Assume $g(t)$ has "compact support",

i.e. $g(t)$ vanishes outside some finite interval.



compact support

$$\int_{-\infty}^{+\infty} f'(t)\,g(t)\,dt = -\int_{-\infty}^{+\infty} f(t)\,g'(t)\,dt$$

Basis for idea of "generalized derivative".

$f(t) = u(t)$ $\quad \underline{\underline{f'(t) = u'(t)}}$

"A function is only as good as how it integrates against other functions"

$$\int_{-\infty}^{+\infty} u'(t)\,g(t)\,dt = -\int_{-\infty}^{+\infty} u(t)\,g'(t)\,dt$$

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 1 \end{cases}$$

$$= -\int_{0}^{+\infty} u(t)\,g'(t)\,dt$$

$$= -\int_{0}^{+\infty} g'(t)\,dt$$

By FTC $\Rightarrow$ $= -\left[g(t)\right]_0^{+\infty}$

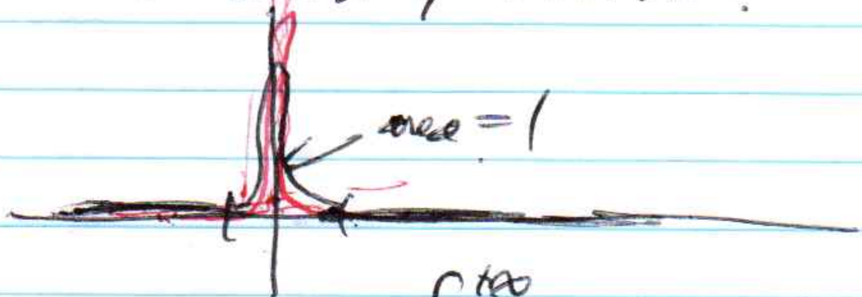
$$= -\left[0 - g(0)\right] = g(0)$$

$u'(t) = \delta(t)$

$$\int_{-\infty}^{+\infty} \delta(t)\, g(t)\, dt = g(0)$$

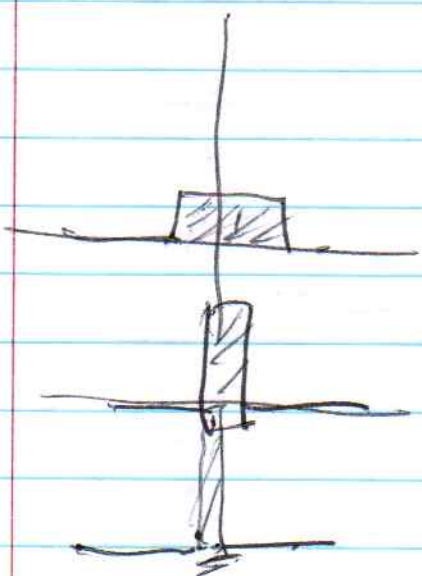
Different perspective of $\delta(t)$.

Probability densities:



$$\int_{-\infty}^{+\infty} p(t)\, dt = 1 \quad \text{probability densities.}$$

$$p(t) \geq 0$$



Remarkable property of $\delta(t)$

$$\boxed{\int_{-\infty}^{+\infty} \delta(t)\, dt} = \int_{-\infty}^{+\infty} u'(t)\, dt$$

$$= u(t) \Big]_{-\infty}^{+\infty} = 1 - 0 = 1$$

etc

probability density.

LAPLACE TRANSFORM

BIG IDEA:

$$\begin{tikzcd}
\boxed{\text{LINEAR ODE w/ IC's}} \arrow[r, "\mathcal{L}"] \arrow[d, "\text{Solve}"] & \boxed{\text{Algebraic Equation}} \arrow[d, "\text{solve}"] \\
\boxed{\text{Response soln to IVP}} \arrow[r, shift left, "\mathcal{L}"] & \boxed{\text{Solution}} \arrow[l, shift left, "\mathcal{L}^{-1}"]
\end{tikzcd}$$

$$\mathcal{L}\left[f(t)\right] \longrightarrow F(s) = \int_{0^-}^{+\infty} e^{-st} f(t)\, dt$$

"Integral Kernel"

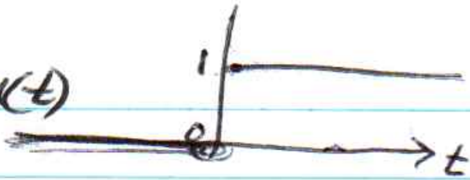
$\mathcal{L}$ is Linear

$$\mathcal{L}\left[a f(t) + b g(t)\right] = \int_{0^-}^{+\infty} e^{-st}\left[a f(t) + b g(t)\right] dt$$

$$= a\int_{0^-}^{+\infty} e^{-st} f(t) + b\int_{0^-}^{\infty} e^{-st} g(t)\, dt$$

$$= a\,F(s) + b\,G(s)$$

$f(t) = 1 \quad = u(t)$



only care about $t > 0$

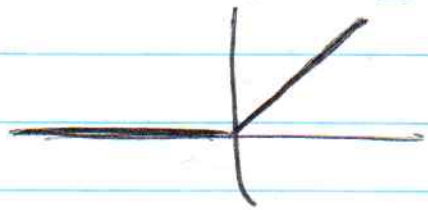
$$\mathcal{L}[1] = \mathcal{L}[u(t)] = \int_{0^-}^{+\infty} e^{-st}\, 1\, dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^{+\infty} = 0 + \frac{1}{s}$$

① $\mathcal{L}[1] = \mathcal{L}[u(t)] = \frac{1}{s}$

Linearity $\mathcal{L}[c] = \mathcal{L}[c\,u(t)] = \frac{c}{s}$

② $f(t) = t \quad = t\,u(t)$



$$\mathcal{L}[t] = \int_{0^-}^{+\infty} t\, e^{-st}\, dt$$

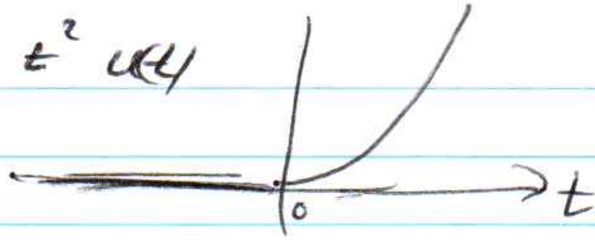
<u>PARTS</u> $\quad u = t \quad dv = e^{-st}\, dt$

$du = dt \quad v = \frac{e^{-st}}{-s}$

$$\cancel{\frac{t\, e^{-st}}{-s}\Big|_0^{+\infty}} + \frac{1}{s}\underbrace{\boxed{\int_0^{+\infty} e^{-st}\, dt}}_{\frac{1}{s}} = \frac{1}{s^2}$$

↑
exponential decay
beats polynomial growth
and Lower Limit $= 0$

$f(t) = t^2$ $\quad$ $t^2 u(t)$



$$\mathcal{L}[t^2] = \int_{0^-}^{+\infty} e^{-st} t^2 \, dt$$

$u = t^2 \quad dv = e^{-st} dt$

$du = 2t\,dt \quad v = \frac{e^{-st}}{-s}$

$$\cancel{\frac{t^2 e^{-st}}{-s}\Bigg|_{0^-}^{+\infty}} + \frac{2}{s}\boxed{\int_0^{+\infty} t e^{-st} \, dt}$$

$$= \frac{2}{s} \cdot \frac{1}{s^2} = \frac{2}{s^3}$$

$\mathcal{L}[t^n]$

s-deriv Rule

If $\mathcal{L}[f(t)] = F(s)$

then

$\mathcal{L}[t\, f(t)] = -F'(s)$

(see notes)

$\mathcal{L}[1] = \frac{1}{s}$

$\mathcal{L}[t] = \frac{1}{s^2}$

$\mathcal{L}[t^2] = \frac{2}{s^3}$

$\mathcal{L}[t^3] = \frac{3!}{s^4} \rightarrow t^3 = t \cdot t^2 \quad \mathcal{L}(t^2) = \frac{2}{s^3}$

$\vdots$

$\mathcal{L}(t^3) = -\frac{d}{ds}\left(\frac{2}{s^3}\right)$

$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$

$= +\frac{3!}{s^4}$

$\mathcal{L}(e^{at}) \qquad e^{at}u(t)$

$$= \int_{0^-}^{+\infty} e^{-st} e^{at}\, dt$$

$$= \int_{0^-}^{+\infty} e^{-(s-a)t}\, dt$$

$$= \left. \frac{e^{-(s-a)t}}{-(s-a)} \right|_{t=0}^{\infty}$$

$$= 0 + \frac{1}{s-a}$$

$$\boxed{\mathcal{L}\left[e^{at}\right] = \frac{1}{s-a}}$$

$\boxed{\delta(t)}$ $\quad \mathcal{L}(\delta(t)) = \int_{0^-}^{+\infty} e^{-st}\delta(t)\,dt$

$$= \int_{-\infty}^{+\infty} e^{-st}\delta(t)\,dt = 1$$

$$\mathcal{L}[\delta(t)] = 1$$

$$\mathcal{L}[\cos wt] \quad , \quad \mathcal{L}[\sin wt]$$

Euler's Identity

$$e^{iwt} = \cos wt + i \sin wt$$

$$\mathcal{L}[e^{iwt}] = \underbrace{\mathcal{L}[\cos wt]}_{\text{Real PART}} + i\underbrace{\mathcal{L}[\sin wt]}_{\text{IMAG PART}}$$

$a = iw$

$$= \frac{1}{s - iw}\left[\frac{s+iw}{s+iw}\right]$$

$$= \left(\frac{s}{s^2+w^2}\right) + i\left(\frac{w}{s^2+w^2}\right)$$

$\mathcal{L}[\cos wt]$ $\qquad$ $\mathcal{L}[\sin wt]$