Problematic Cases

[much of this lecture was done in conjunction with the Lecture #4 and Lecture #5 notes posted on course website]

Solve $\ddot{x} + 4\dot{x} + 4x = 0$

Try $\boxed{x = e^{rt}}$ $\dot{x} = re^{rt}$ $\ddot{x} = r^2 e^{rt}$

$\underbrace{(r^2 + 4r + 4)} e^{rt} = 0$ (REPEATED ROOTS OF CHAR. POLYN)

$p(r) = r^2 + 4r + 4 = (r+2)(r+2) = 0$

$(r+2)^2 = 0$ $r + 2 = 0$

$\boxed{r = -2}$ $AM = 2$

$\text{Span}\{e^{-2t}\}$

Q: Does this yield all solns? $x_h(t) = ce^{-2t}$

A: NO! What do we do?

OPERATORS $(D^2 + 4D + 4I)\, x(t) = 0$

$$\left[(D + 2I) \circ \underbrace{(D + 2I)]\, x(t)}_{\text{Call this } y(t)} = 0$$

$$\frac{dy}{dt} + 2y = 0 \Rightarrow \boxed{y(t) = c_1 e^{-2t}}$$

$$\frac{dx}{dt} + 2x = c_1 e^{-2t}$$ Seek integrating factor

$$e^{\int 2\,dt} = e^{2t}$$

$$\underbrace{e^{2t}\frac{dx}{dt} + 2e^{2t}x}_{\frac{d}{dt}\left[e^{2t}x\right]} = c_1$$

$\Rightarrow e^{2t}x = c_1 t + c_2$

$$\boxed{x(t) = c_1 te^{-2t} + c_2 e^{-2t}}$$

$\text{Span}\{e^{-2t}, te^{-2t}\}$

Independence of solutions and unique solvability of IVP's.

WRONSKIAN
$$\det\begin{bmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & (1-2t)e^{-2t} \end{bmatrix} = e^{-4t}\left[1-2t+2t\right]$$
$$= e^{-4t} \neq 0 \quad \checkmark$$
all t

$$\frac{d}{dt}\left[te^{-2t}\right] = -2te^{-2t} + e^{-2t} = e^{-2t}(1-2t)$$

<u>Complex Roots.</u> $\qquad \omega \neq 0$

$$\ddot{x} + \omega^2 x = 0 \qquad x = e^{rt}$$
$$p(r) = r^2 + \omega^2 = 0 \qquad r = \pm i\omega$$
$$\text{Span}\left\{e^{i\omega t}, e^{-i\omega t}\right\}$$
$$x(t) = c_1 e^{i\omega t} + c_2 e^{-i\omega t}$$
(Formally correct, but requires complex coefficients.)

<u>WRONSKIAN:</u>
$$\det\begin{vmatrix} e^{i\omega t} & e^{-i\omega t} \\ i\omega e^{i\omega t} & -i\omega e^{-i\omega t} \end{vmatrix}$$
$$= -i\omega - i\omega = -2i\omega \neq 0 \quad \checkmark$$

<u>Alternative:</u>
$$e^{i\omega t} = \cos\omega t + i\sin\omega t$$
$$e^{-i\omega t} = \cos\omega t - i\sin\omega t$$
$$x(t) = c_1(\cos\omega t + i\sin\omega t) + c_2(\cos\omega t - i\sin\omega t)$$
$$= \underbrace{(c_1 + c_2)}\cos\omega t + i\underbrace{(c_1 - c_2)}\sin\omega t$$
$$= b_1\cos\omega t + b_2\sin\omega t$$
(Real coefficients)

$$\text{Span}\left\{\cos\omega t, \sin\omega t\right\}$$

WRONSKIAN

$$\det\begin{bmatrix} \cos\omega t & \sin\omega t \\ -\omega\sin\omega t & \omega\cos\omega t \end{bmatrix}$$

$$= \omega\left(\underbrace{\cos^2\omega t + \sin^2\omega t}_{=1}\right) = \omega \neq 0$$

by Pythagorean Identity