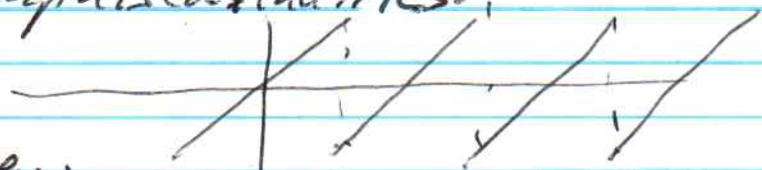
PERIODIC INPUTS + FOURIER SERIES

Period T $\quad f(t+nT)=f(t)$

$\cos t$, $\sin t$, $\cos 2t$, $\sin 2t$ ...
constant function.

Theorem (FOURIER): If $f(t)$ is periodic with base period 2π, continuous except for a finite number of jump discontinuities.



then $f(t)$ ~~may~~ be represented by a convergent Fourier Series of form

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt)$$

where $a_0 = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(t)\,dt \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(t)\cos nt\,dt$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(t)\sin nt\,dt$$

$\{a_0, a_1, b_1, a_2, b_2, \ldots\} \Rightarrow$ Fourier coefficients

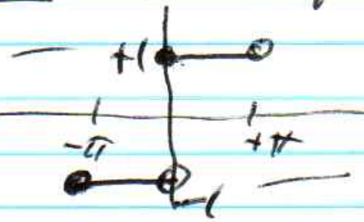
Note 1: If $f(t)$ an even function $[f(-t)=f(t)]$, all $b_n=0$.

Note 2: If $f(t)$ is an odd function $[f(-t)=-f(t)]$, then
$a_0=0$ and all $a_n=0$.

Note 3: $\frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(t)\,dt = \bar{f}$ avg value of f

FS Calculations $\Leftrightarrow$ Integration by Parts

__Ex:__ $f(t) = Sq(t)$ = "Square wave function"



$$Sq(t) = \begin{cases} -1 & [-\pi, 0) \\ +1 & (0, \pi) \end{cases}$$

extend periodically

f odd function, all $a_0 = 0$, $a_n = 0$.

$$b_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(t) \sin nt \, dt$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{0} (-1) \sin nt \, dt + \int_0^{\pi} (+1) \sin nt \, dt \right]$$

$$= \frac{1}{\pi} \left[\left[\frac{\cos nt}{n} \right]_{-\pi}^{0} - \left[\frac{\cos nt}{n} \right]_0^{\pi} \right]$$

$$= \frac{1}{n\pi} \left[\left(1 - (-1)^n\right) - \left((-1)^n - 1\right) \right]$$

$$= \begin{cases} \frac{4}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

$$Sq(t) \sim \sum_{n \text{ odd}} \frac{4}{n\pi} \sin nt = \frac{4}{\pi} \left(\sum_{n \text{ odd}} \frac{\sin nt}{n} \right)$$

$$= \frac{4}{\pi} \left[\sin t + \frac{\sin 3t}{3} + \frac{\sin 5t}{5} + \cdots \right]$$

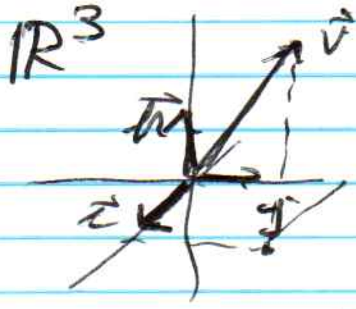
__Note 4:__ At any point of continuity, FS converges to value $f(t)$

At any jump discontinuity $t = a$, FS converges to average $\dfrac{f(a^-) + f(a^+)}{2}$.

Where does FS come from?

Inner Products and ORTHONORMAL Function.

$\mathbb{R}^3$



ORTHONORMAL basis for $\mathbb{R}^3$ $\{\vec{i}, \vec{j}, \vec{k}\}$

$$\vec{V} = \langle v_x, v_y, v_z \rangle$$
$$= \langle v_x, 0, 0\rangle + \langle 0, v_y, 0\rangle + \langle 0, 0, v_z\rangle$$
$$= v_x \vec{i} + v_y \vec{j} + v_z \vec{k}$$

DOT PRODUCT

$$= (\vec{V}\cdot\vec{i})\vec{i} + (\vec{V}\cdot\vec{j})\vec{j} + (\vec{V}\cdot\vec{k})\vec{k}$$

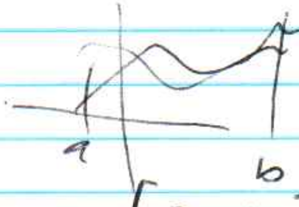
$$v_x = \vec{V}\cdot\vec{i}$$
$$v_y = \vec{V}\cdot\vec{j}$$
$$v_z = \vec{V}\cdot\vec{k}$$

In a space of functions, similar CONSTRUCTION.

$$\vec{u}\cdot\vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 + \cdots + u_n v_n$$
$$= u(1)v(1) + u(2)v(2) + \cdots + u(n)v(n)$$

Continuous or piecewise continuous fn on $[a, b]$

$$\langle f, g\rangle = \int_a^b f(t)g(t)\,dt \quad \left(\text{analogous to dot product}\right)$$



More generally and more specifically

$$\langle f, g\rangle = K\int_{-\pi}^{+\pi} f(t)g(t)\,dt$$

on $[-\pi, +\pi]$

K any positive constant

DOT PRODUCT properties

$\vec{v}\cdot\vec{u} = \vec{u}\cdot\vec{v}$

$\vec{v}\cdot(\vec{u}+\vec{w}) = \vec{v}\cdot\vec{u} + \vec{v}\cdot\vec{w}$

$(\vec{v}+\vec{u})\cdot\vec{w} = \vec{v}\cdot\vec{w} + \vec{u}\cdot\vec{w}$

$(t\vec{u})\cdot\vec{v} = t(\vec{u}\cdot\vec{v}) = \vec{u}\cdot t\vec{v}$

$\vec{u}\cdot\vec{u} = \|\vec{u}\|^2 \geq 0$ $= 0$ only if $\vec{u} = 0$

Inner Product

$\langle g, f\rangle = \langle f, g\rangle$

$\langle f, g+h\rangle = \langle f, g\rangle + \langle f, h\rangle$

$\langle f+g, h\rangle = \langle f, h\rangle + \langle g, h\rangle$

$\langle t f, g\rangle = t\langle f, g\rangle$

$\langle f, f\rangle = \|f\|^2 \geq 0$

The collection $\left\{\frac{1}{\sqrt{2}}, \cos t, \sin t, \cos 2t, \sin 2t, \dots, \cos nt, \sin nt\right\}$

is an ORTHONORMAL SET.

f, g orthogonal if $\langle f, g\rangle = 0$

$\|f\| = 1$ $\quad 2n+1$ of these.

Set spans a subspace of dimension $2n+1$

$$\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = \frac{1}{\pi}\int_{-\pi}^{+\pi}\left(\frac{1}{2}\right)dt = \frac{1}{\pi}\,\frac{1}{2}t\Big]_{-\pi}^{\pi} = 1$$

$$\boxed{\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(t)g(t)\,dt}$$

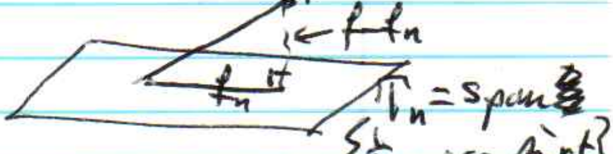
$$\left\|\frac{1}{\sqrt{2}}\right\| = 1$$

$$\left\langle \frac{1}{\sqrt{2}}, \cos nt \right\rangle = \frac{1}{\pi}\int_{-\pi}^{+\pi}\frac{1}{\sqrt{2}}\cos nt\,dt = \frac{1}{\pi\sqrt{2}}\left[\frac{\sin nt}{n}\right]_{-\pi}^{\pi} = 0$$

see lecture notes for remaining ORTHOGONALITY RELATIONS.

f, f_n nth order Fourier approximation

$$f = \underbrace{(f - f_n)}_{\text{orthogonal to } f_n} + \underbrace{f_n}_{\in \operatorname{span}\{\frac{1}{\sqrt{2}}, \dots, \sin nt\}}$$



Pythagorean theorem $\|f\|^2 = \|f - f_n\|^2 + \|f_n\|^2$

FS convergence means that

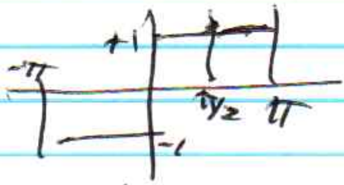
$$\lim_{n\to\infty} \|f - f_n\| = 0$$

$$\Rightarrow \boxed{\lim_{n\to\infty} \|f_n\|^2 = \|f\|^2} \Rightarrow \text{Interesting FACTS}$$

$$Sf(t) \sim \frac{4}{\pi}\left(\sum_{n \text{ odd}} \frac{\sin nt}{n}\right)$$

$$= \frac{4}{\pi}\left[\sin t + \frac{\sin 3t}{3} + \cdots + \frac{\sin(2n+1)t}{2n+1}\right.$$

Silly observation



$t = \frac{\pi}{2}$ is a point of continuity

$$\frac{4}{\pi}\left(\sin\frac{\pi}{2} + \frac{1}{3}\sin\frac{3\pi}{2} + \frac{1}{5}\sin\frac{5\pi}{2} + \cdots\right) = 1$$

$$4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots\right) = \pi$$

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots = \frac{\pi}{4}$$

conditional convergence, extremely slow convergence

Note 5: By orthogonality of all the terms.

$$\frac{a_0^2}{2} + a_1^2 + b_1^2 + a_2^2 + b_2^2 + \cdots + a_n^2 + b_n^2 = \|f_n\|^2$$

$$\lim_{n\to\infty} \|f_n\|^2 = \|f\|^2$$

$$\frac{a_0^2}{2} + a_1^2 + b_1^2 + \cdots + a_n^2 + b_n^2 + \cdots = \|f\|^2$$

Source of entertaining curiosities.

Ex: $Sf(t) = \frac{4}{\pi}\left[\sin t + \frac{1}{3}\sin 3t + \cdots + \frac{1}{2n+1}\sin(2n+1)t + \cdots\right.$

$a_0 = 0$, all $a_n = 0$

$$b_1^2 + b_2^2 + \cdots b_n^2 + \cdots = \|f\|^2$$

$$\frac{16}{\pi^2}\left[1 + \frac{1}{9} + \frac{1}{25} + \cdots + \frac{1}{(2n+1)^2} + \cdots\right) = \|f\|^2$$

$f(t) = \operatorname{sgn}(t)$

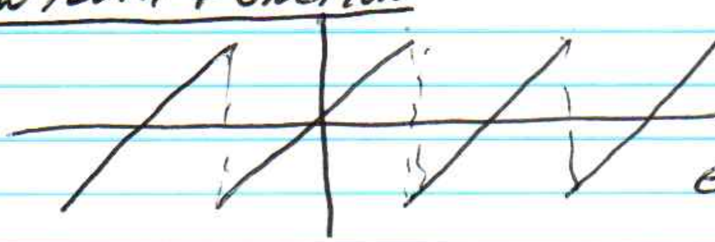
$$\|f\|^2 = \langle f, f \rangle = \frac{1}{\pi}\int_{-\pi}^{+\pi} 1\,dt = \frac{1}{\pi}(2\pi) = 2$$

$$\frac{16}{\pi^2}\left[\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}\right] = 2$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$$

$$1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \cdots = \frac{\pi^2}{8}.$$

SAWTOOTH FUNCTION



$f(t) = t \ [-\pi, +\pi]$
extend periodically

ODD FUNCTION $a_0 = 0$, all $a_n = 0$

$$b_n = \langle f, \sin nt \rangle = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(t) \sin nt\,dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{+\pi} t \sin nt\,dt \qquad \text{LIPET}$$

Integration by Parts $\left[\begin{array}{ll} u = t & dv = \sin nt\,dt \\ du = dt & v = -\frac{\cos nt}{n} \end{array}\right]$

$$= \frac{1}{\pi}\left[-\frac{t\cos nt}{n}\Big|_{-\pi}^{\pi} + \frac{1}{n}\int_{-\pi}^{+\pi} \cos nt\,dt \right]$$

$$= \frac{2}{\pi}\left[-\frac{\pi(-1)^n}{n} + 0 \right] = (-1)^{n+1} \cdot \frac{2}{n}$$

$$f(t) \sim 2\left[\sum_{n=1}^{\infty} (-1)^n \frac{\sin nt}{n}\right]$$

$$= 2\left[-\sin t + \frac{\sin 2t}{2} - \frac{\sin 3t}{3} + \cdots \right]$$

In term of norm $\|f\|$

$$\|f\|^2 = \sum_{n=1}^{\infty} b_n^2 = \sum_{n=1}^{\infty} \frac{4}{n^2}$$

$$\langle f,f\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2\,dt = \frac{1}{\pi}\frac{t^3}{3}\Big|_{-\pi}^{+\pi} = \frac{2\pi^2}{3}$$

$f(t)=t$ on $[-\pi,\pi)$

$$\therefore \sum_{n=1}^{\infty} \frac{4}{n^2} = \frac{2\pi^2}{3}$$

$$\Rightarrow \boxed{\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}}$$

p-series $p=2$

Solve $\ddot{x} + 3x = f(t)$ - Sawtooth function

Rep $f(t)$ as FS $\Rightarrow f(t) \sim 2\sum_{n=1}^{\infty} (-1)^n \frac{\sin nt}{-n}$

BIG IDEA: Solve ODE term by term, Reassemble

nth term $\ddot{x} + 3x = \dfrac{2(-1)^n \sin nt}{n}$

Complex Replacement and ERF

$$\ddot{z} + 3z = \frac{2(-1)^n}{n} e^{int} \quad \text{Im part}$$

ERF: $p(s) = s^2+3 \quad p(in) = 3-n^2$

$$z_n = \frac{2(-1)^n}{n}\frac{e^{int}}{p(in)} = \frac{2(-1)^n(\cos nt + i\sin nt)}{n(3-n^2)}$$

[See notes for Resonance example]

"Imaginary" Part $\quad x_n = \dfrac{2(-1)^n}{n(3-n^2)}\sin nt$

reassembled

$$\therefore x_p(t) = \sum_{n=1}^{\infty} \frac{2(-1)^n}{n(3-n^2)} \sin nt$$