UNIT IMPULSE RESPONSE

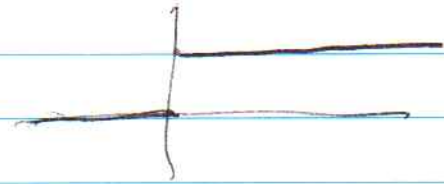
$[p(D)]\,x(t) = \delta(t)$ Rest Initial Conditions.

$\ddot{x} + 3\dot{x} + 2x = \delta(t)$ $x(0)=0,\ \dot{x}(0)=0$

Laplace Transform

$$\downarrow \mathcal{L}$$

$$(s^2+3s+2)\,X(s) = 1$$



$$p(s)\,X(s) = 1$$

$$X(s) = \frac{1}{p(s)} = \frac{1}{s^2+3s+2} = \frac{1}{(s+2)(s+1)} = \frac{A}{s+2} + \frac{B}{s+1}$$

$$1 = A(s+1) + B(s+2)$$

$s=-1$ $1 = B$

$s=-2$ $1 = -A$ $A = -1$

$$X(s) = \frac{1}{s+1} - \frac{1}{s+2} = W(s) \quad \left(\text{TRANSFER FUNCTION}\right)$$

$$\downarrow \mathcal{L}^{-1}$$

$$x(t) = e^{-t} - e^{-2t} = w(t)$$

(weight function)

SITUATION: Solve $[p(D)]\,x(t) = f(t)$ (*) $x(0)=x_0$, $\dot{x}(0)=\dot{x}_0$ etc

ZSR + ZIR [Zero state Response + Zero Input Response]

① ZSR [zero state Response]

Solve $[p(D)]\,x(t) = f(t)$, Rest IC's $x(0)=0$, $\dot{x}(0)=0$ etc

$$\Rightarrow \text{ZSR} = x_{ZSR}(t)$$

② ZERO INPUT RESPONSE (ZIR)

solve $[p(D)]x(t) = 0$; $x(0) = x_0$, $\dot{x}(0) = \dot{x}_0$, etc

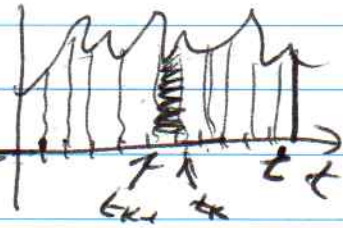
$\Rightarrow x_{ZIR}(t)$

"Two Wrongs make a Right"

$x(t) = x_{ZSR}(t) + x_{ZIR}(t)$ solves ⊛

<u>CONVOLUTION</u> ⊛ $[p(D)]x(t) = f(t)$

IC $x(0) = x_0$, $\dot{x}(0) = \dot{x}_0$, etc.



$f(t)$ might be given by data, Real-time signal.

<u>IDEA:</u> $\boxed{\text{① Solve ZSR first. ⊛, Rest IC's.}}$

② solve ZIR $[p(D)]x(t) = 0$, correct IC's.

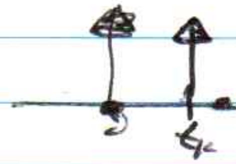
<u>STEP-BY-STEP</u>

① Solve unit Impulse Response $[p(D)]x(t) = \delta(t)$ Rest IC's,

$\Rightarrow w(t)$ weight function.

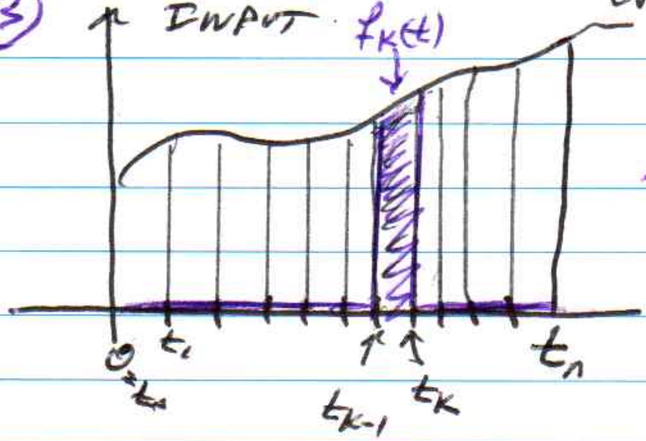
② <u>TIME-INVARIANCE</u>

If $w(t)$ solves $[p(D)]x(t) = \delta(t)$



Then $w(t-t_k)$ solves $[p(D)]x(t) = \delta(t-t_k)$

③



PARTITION $[0, t]$

$f_k(t) = f(t)\left[u(t_{k-1}) - u(t_k)\right]$

<u>Note:</u> $\sum_{k=1}^{n} f_k(t) = f(t)$

note: on $[t_{k-1}, t_k]$

$$f_k(t) \doteq f(t_k) = \int_{-\infty}^{+\infty} f(t)\,\delta(t-t_k)\,dt$$
$$= \int_0^{+\infty} f(t)\,\delta(t-t_k)\,dt$$

(4) soln to $[p(D)]x(t) = \delta(t-t_k)$ is $w(t-t_k)$

INVOKE LINEARITY

Soln to $[p(D)]x(t) = f(t_k)\,\delta(t-t_k)\,\Delta t_k$

is $\boxed{f(t_k)\,w(t-t_k)\,\Delta t_k}$

where $\Delta t_k = t_k - t_{k-1}$

(5) Superposition

soln to $[p(D)]x(t) = \sum_{k=1}^{n} f(t_k)\,\delta(t-t_k)\,\Delta t_k$

is $\sum_{k=1}^{n} f(t_k)\,w(t-t_k)\,\Delta t_k$
$= \sum_{k=1}^{n} f(\tau_k)\,w(t-\tau_k)\,\Delta \tau_k.$

(6) Calculus Let $|\Delta| \to 0$

soln to $[p(D)]x(t) = \int_0^t f(\tau)\,\delta(t-\tau)\,d\tau$
$= f(t)$

will be $\int_0^t f(\tau)\,w(t-\tau)\,d\tau = (f*w)(t)$

$\boxed{\text{Convolution Integral}}$

CONVOLUTION PRODUCT.

Note: $(f*w)(t) = (w*f)(t)$
$= \int_0^t f(t-\tau)\,w(\tau)\,d\tau$

Summary: Soln to $[p(D)]\,x(t) = f(t)$

w/ zero state initial conditions

is $(f*w)(t) = \int_0^t f(\tau)\,w(t-\tau)\,d\tau$

$\Rightarrow$ ZSR

SIDE NOTE: $\mathcal{L}(f(t)) = F(s)$, $\mathcal{L}(g(t)) = G(s)$

then $\mathcal{L}[(f*g)(t)] = F(s)G(s)$

EXAMPLE: Solve $\ddot{x} + 3\dot{x} + 2x = 2e^{-t}$

IC's: $x(0) = 4$, $\dot{x}(0) = 0$

OLD FASHIONED SOLN: Homog $\ddot{x} + 3\dot{x} + 2x = 0$

$p(s) = s^2 + 3s + 2 = (s+2)(s+1)$

$x_h(t) = c_1 e^{-t} + c_2 e^{-2t}$

PART: ERF $x_p(t) = \dfrac{2e^{-t}}{p(-1)}$ FAILS!

RRF $x_p(t) = \dfrac{2te^{-t}}{p'(-1)} = \dfrac{2te^{-t}}{1} = 2te^{-t}$

$p'(s) = 2s + 3$

$p'(-1) = -2 + 3 = 1$

$x(t) = c_1 e^{-t} + c_2 e^{-2t} + 2t e^{-t}$ general SOLN

$\dot{x}(t) = -c_1 e^{-t} - 2c_2 e^{-2t} + 2e^{-t} - 2te^{-t}$

$x(0) = c_1 + c_2 = 4$

$\dot{x}(0) = -c_1 - 2c_2 + 2 = 0$

$\rightarrow c_1 = 6$, $c_2 = -2$

$x(t) = 6e^{-t} - 2e^{-2t} + 2te^{-t}$ ✓

TRANSFORM, CONVOLUTION, ZSR + ZIR

Solve $\ddot{x} + 3\dot{x} + 2x = 2e^{-t}$ $\quad x(0) = 4 \quad \dot{x}(0) = 0$

UNIT IMPULSE RESPONSE $\quad$ WEIGHT FUNCTION

$$\ddot{x} + 3\dot{x} + 2x = \delta(t) \quad \text{REST IC's}$$

$$\Rightarrow \boxed{w(t) = e^{-t} - e^{-2t}}$$

INPUT $\quad f(t) = 2e^{-t}$

$$ZSR = \int_0^t f(\tau)\, w(t-\tau)\, d\tau = (f * w)(t)$$

$$\left[= \int_0^t w(\tau)\, f(t-\tau)\, d\tau \right]$$

$$= \int_{\tau=0}^{\tau=t} 2e^{-\tau}\left[e^{-(t-\tau)} - e^{-2(t-\tau)}\right] d\tau$$

$$= \int_0^t \left(2e^{-t} - 2e^{-2t}e^{\tau}\right) d\tau$$

$$= 2e^{-t}\int_{\tau=0}^{\tau=t} 1\, d\tau - 2e^{-2t}\int_{\tau=0}^{\tau=t} e^{\tau}\, d\tau$$

$$= 2e^{-t}\cdot t - 2e^{-2t}\left[e^{\tau}\right]_{\tau=0}^{\tau=t}$$

$$= 2te^{-t} - 2e^{-2t}\left[e^{t} - 1\right]$$

$$= 2te^{-t} - 2e^{-t} + 2e^{-2t}$$

$$\boxed{ZSR = -2e^{-t} + 2e^{-2t} + 2te^{-t}}$$

Zero Input Response (ZIR)

Solve $\ddot{x} + 3\dot{x} + 2x = 0$, $x(0) = 4$, $\dot{x}(0) = 0$

$$p(s) = s^2 + 3s + 2$$
$$= (s+1)(s+2)$$

$$x(t) = c_1 e^{-t} + c_2 e^{-2t}$$
$$\dot{x}(t) = -c_1 e^{-t} - 2c_2 e^{-2t}$$

$$x(0) = c_1 + c_2 = 4 \qquad c_2 = -4$$
$$\dot{x}(0) = -c_1 - 2c_2 = 0 \qquad c_1 = 8$$
$$-c_2 = 4$$

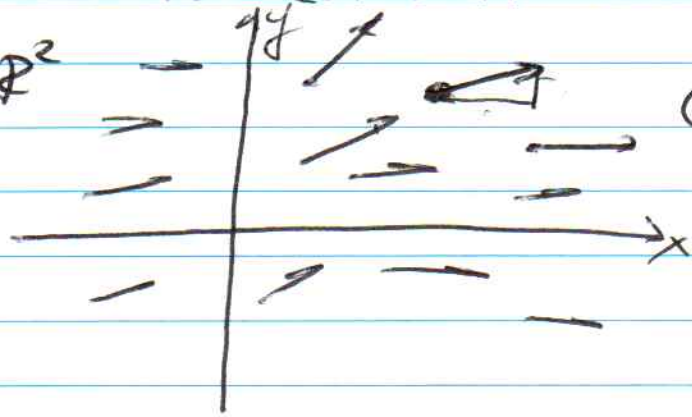
$$\boxed{ZIR = 8e^{-t} - 4e^{-2t}}$$

$$x(t) = ZSR + ZIR$$
$$= -2e^{-t} + 2e^{-2t} + 2te^{-t}$$
$$+ 8e^{-t} - 4e^{-2t}$$
$$= 6e^{-t} - 2e^{-2t} + 2te^{-t} \checkmark$$

Vector fields, continuous dynamical systems, systems of 1st order ODE's.

Autonomous (time-invariant) vector fields

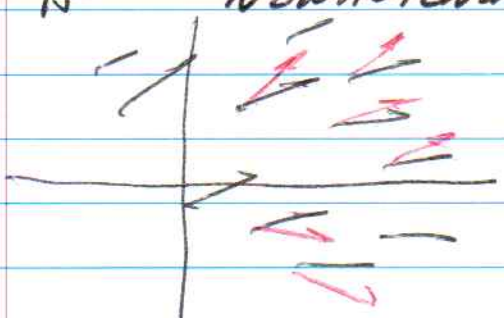
$\mathbb{R}^2$



$$(x,y) \to \vec{F}(x,y) = \langle F_1(x,y), F_2(x,y) \rangle$$

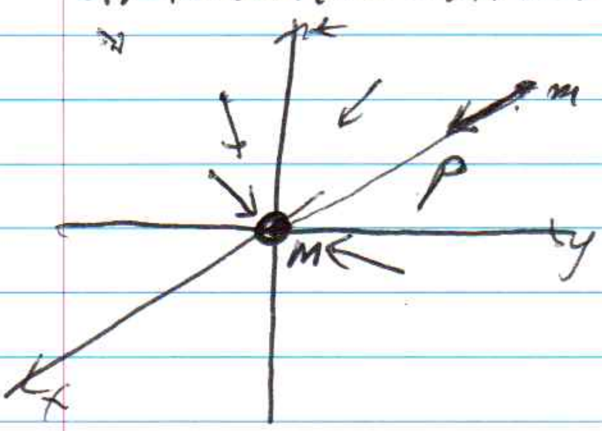
STATIC VECTOR FIELD

$\mathbb{R}^2$ NONAUTONOMOUS VECTOR FIELD



$$(x,y,t) \to \vec{F}(x,y,t) = \langle F_1(x,y,t), F_2(x,y,t) \rangle$$

GRAVITATIONAL ATTRACTION

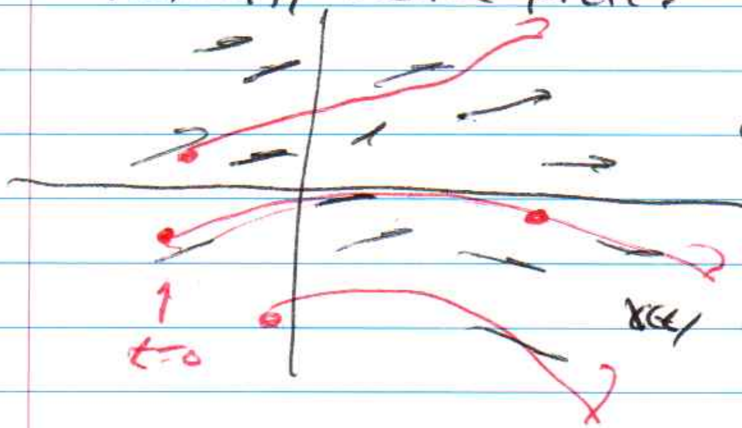


$$\boxed{\vec{F} = -\left(\frac{GMm}{\rho^2}\right) \vec{u}_\rho}$$

$\vec{u}_\rho$ = unit Radial Vector

singularity at $\rho = 0$.

velocity vector field.



$(x,y) \to \vec{V}(x,y)$

Initial conditions $(x(0), y(0))$

$$\frac{d\vec{x}}{dt} = \vec{V} = (F_1(x_1,\dots,x_n), \dots, F_n(x_1,\dots,x_n))$$

$\underline{\underline{x(t)}}$ solution curve
integral curve
trajectory
orbit

1st ORDER SYSTEM of ODES. (AUTONOMOUS)

$$\left\{\begin{aligned}
\frac{dx_1}{dt} &= F_1(x_1,\dots,x_n) \\
\frac{dx_2}{dt} &= F_2(x_1,\dots,x_n) \\
\frac{dx_3}{dt} &= \\
&\vdots \\
\frac{dx_n}{dt} &= F_n(x_1,\dots,x_n)
\end{aligned}\right\}$$

nth order ODE $\longrightarrow$ system of 1st order ODEs

Reduction of order

2nd order $\boxed{\ddot{x} + 3\dot{x} + 2x = 0}$ (homog)

$$\frac{dx}{dt} = \dot{x} = y$$

$$\frac{d^2x}{dt^2} = \ddot{x} = \dot{y} = \frac{dy}{dt} = -2x - 3y$$

$$\left\{\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -2x - 3y \end{aligned}\right\} \checkmark \quad A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

Ex: $\ddot{x} + 3t^2\dot{x} + 2e^t x = 0$

$$\left\{\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -2e^t x - 3t^2 y \end{aligned}\right\}$$ nonautonomous vector field

Ex: $\boxed{\dddot{x} + 3\ddot{x} - 4\dot{x} + 2x = 0}$

$$\begin{array}{r} \dot{x} \\ \ddot{x} = \\ \dddot{x} = \end{array} \boxed{\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= z \\ \frac{dz}{dt} &= 2x + 4y - 3z \end{aligned}} \quad \checkmark \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 4 & -3 \end{bmatrix}$$

SPECIAL CASE:

$$\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n$$

$$\frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n$$

$$\vdots \qquad\qquad \vdots$$

$$\frac{dx_n}{dt} = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n$$

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \qquad \frac{d\vec{x}}{dt} = \begin{bmatrix} \frac{dx_1}{dt} \\ \vdots \\ \frac{dx_n}{dt} \end{bmatrix} = \underbrace{\begin{bmatrix} a_{11} & \cdots & & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & \cdots & & a_{nn} \end{bmatrix}}_{\substack{A \\ n \times n \\ \text{matrix}}} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$\boxed{\frac{d\vec{x}}{dt} = A\vec{x}}$$

Always have explicit Solutions

SIMPLEST CASE

$$\frac{dx}{dt} = kx, \quad x(0) = x_0$$

$$\int \frac{dx}{x} = \int k\,dt \qquad \ln|x| = kt + C$$

$$e \qquad e$$

$$x(t) = a e^{kt} \qquad x(0) = a$$

$$\boxed{x(t) = e^{kt} x(0)}$$

Uncoupled system

$$\left\{ \begin{array}{l} \frac{dx}{dt} = k_1 x \\ \frac{dy}{dt} = k_2 y \end{array} \right\} \qquad \begin{array}{l} x(0) = x_0 \\ y(0) = y_0 \end{array}$$

$$D = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}$$

$$\left\{ \begin{array}{l} x(t) = x_0 e^{k_1 t} \\ y(t) = y_0 e^{k_2 t} \end{array} \right\} \qquad \boxed{\frac{d\vec{x}}{dt} = D\vec{x}}, \ \vec{x}(0)$$

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} x_0 e^{k_1 t} \\ y_0 e^{k_2 t} \end{bmatrix} = \begin{bmatrix} e^{k_1 t} & 0 \\ 0 & e^{k_2 t} \end{bmatrix} \begin{bmatrix} x(0) \\ y(0) \end{bmatrix}$$

$$\left[e^{tD} \right] \quad \text{evolution MATRIX.}$$

$$\Rightarrow \vec{x}(t) = \left[e^{tD} \right] \vec{x}(0)$$

Generally: $\frac{d\vec{x}}{dt} = A\vec{x}, \ \vec{x}(0) \longrightarrow \vec{x}(t) = \left[e^{tA} \right] \vec{x}(0)$