

## Reminder: Changing coordinates

$$\begin{array}{ccc} \{\mathbb{R}^n, \mathcal{E}\} & \xrightarrow{A} & \{\mathbb{R}^n, \mathcal{E}\} \\ \uparrow S & & \uparrow S \\ \{\mathbb{R}^n, \mathcal{B}\} & \xrightarrow{[A]_{\mathcal{B}}} & \{\mathbb{R}^n, \mathcal{B}\} \\ [\vec{x}]_{\mathcal{B}} & & \end{array} \quad \begin{array}{l} \mathcal{E} = \{\vec{e}_1, \dots, \vec{e}_n\} \\ \text{STANDARD BASIS} \\ \\ \mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\} \\ \text{alternate basis} \end{array}$$

If  $\vec{x}$  (standard coords) is expressed as

$$\vec{x} = c_1 \vec{v}_1 + \dots + c_n \vec{v}_n \quad \text{in other basis, so } \{c_1, \dots, c_n\} \text{ are coords relative to basis } \mathcal{B}.$$
$$\vec{x} = \begin{bmatrix} \vec{v}_1 & \dots & \vec{v}_n \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

$$\vec{x} = S [\vec{x}]_{\mathcal{B}}$$

FROM DIAGRAM ABOVE

$$[A]_{\mathcal{B}} = S^{-1} A S = B$$

or  $A = S B S^{-1}$

We generally seek a basis  $\mathcal{B}$  consisting of eigenvectors and other linearly independent vectors so that the matrix  $B$  is simple (standardized), e.g. in DIAGONAL FORM in the case of Real, distinct eigenvalues.

For a system of ODE's of form  $\frac{d\vec{x}}{dt} = A\vec{x}$ ,  $\vec{x}(0)$ ,

in new coordinates  $\frac{d\vec{x}}{dt} = S B S^{-1} \vec{x}$   $\vec{u} = S^{-1} \vec{x} = [\vec{x}]_{\mathcal{B}}$

$$\Rightarrow S^{-1} \frac{d\vec{x}}{dt} = \frac{d}{dt} (S^{-1} \vec{x}) = B (S^{-1} \vec{x}) \Rightarrow \frac{d\vec{u}}{dt} = B \vec{u}$$

If  $[e^{tB}]$  can be easily determined, then  $\vec{u}(t) = [e^{tB}] \vec{u}(0)$

$$\Rightarrow S^{-1} \vec{x} = [e^{tB}] S^{-1} \vec{x}(0) \Rightarrow \vec{x}(t) = S [e^{tB}] S^{-1} \vec{x}(0)$$

$$\text{so } \vec{x}(t) = \underbrace{[e^{tA}]}_{\text{solves system}} \vec{x}(0)$$

$$\begin{cases} \frac{dx}{dt} = 5x - 6y \\ \frac{dy}{dt} = 3x - 4y \end{cases}$$

$$x(0) = 3, y(0) = 1$$

$$\frac{d\vec{x}}{dt} = A\vec{x} \quad A = \begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix}$$

$$\lambda I - A = \begin{bmatrix} \lambda - 5 & 6 \\ -3 & \lambda + 4 \end{bmatrix}$$

$$\vec{x}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$P_A(\lambda) = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1) = 0$$

See  
Lecture Notes #12

$$\lambda_1 = 2 \quad \lambda_2 = -1$$

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$S = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \quad S^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

$$S^{-1}AS = D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\frac{d\vec{x}}{dt} = A\vec{x}$$

$$A = SDS^{-1}$$

$$\frac{d\vec{x}}{dt} = SDS^{-1}\vec{x}$$

$$\frac{d}{dt}(S^{-1}\vec{x}) = D(S^{-1}\vec{x})$$

$$\boxed{\frac{d\vec{u}}{dt} = D\vec{u}}$$

$$\vec{u}(t) = \begin{bmatrix} e^{2t} \\ e^{-t} \end{bmatrix} \vec{u}(0)$$

$$\vec{u} = S^{-1}\vec{x}$$

$$S^{-1}\vec{x} = \begin{bmatrix} e^{2t} \\ e^{-t} \end{bmatrix} S^{-1}\vec{x}(0)$$

$$\vec{x} = S\vec{u}$$

$$\vec{x}(t) = S \begin{bmatrix} e^{2t} \\ e^{-t} \end{bmatrix} S^{-1}\vec{x}(0)$$

$\underbrace{\begin{bmatrix} e^{2t} \\ e^{-t} \end{bmatrix}}_{[e^{tA}]}$  evolution matrix

$$\frac{d\vec{x}}{dt} = A\vec{x}$$

$$S^{-1}AS = D$$

$$A = SDS^{-1}$$

$$[e^{tA}] = S[e^{tD}]S^{-1}$$

$$\vec{x}(t) = [e^{tA}] \vec{x}(0) = S[e^{tD}]S^{-1} \vec{x}(0)$$

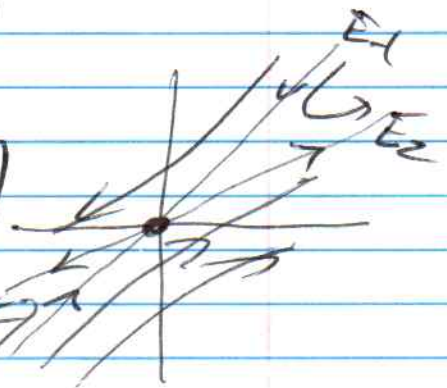
$$\vec{x}(t) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{2t} & e^{-t} \\ e^{2t} & e^{-t} \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \quad \boxed{\text{See APLANE}}$$

$$\vec{x}(t) = \begin{bmatrix} 4e^{2t} - e^{-t} \\ 2e^{2t} - e^{-t} \end{bmatrix} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

$$= 2e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} - e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\uparrow$   $\vec{v}_1$                        $\uparrow$   $\vec{v}_2$



$$\left\{ \begin{array}{l} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -\omega^2 x \end{array} \right\}$$

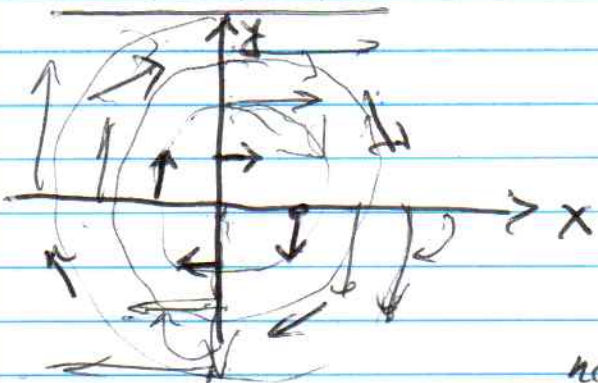
$\ddot{x} + \omega^2 x = 0$   
HARMONIC OSCILLATOR

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = -\omega^2 x$$

$$\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix} \quad \frac{d\vec{x}}{dt} = A\vec{x} \quad A = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix}$$

## Vector Field:



NO INVARIANT  
DIRECTIONS

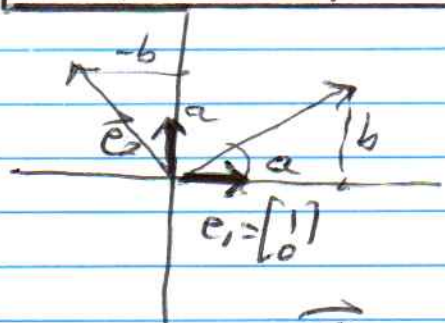
Real  
No eigenvectors

no Real eigenvalues

## Algebraic Formalism.

$S^{-1}AS = B$  still works.

### ASIDE: ROTATION-DILATION MATRIX.



$$\begin{aligned} \vec{e}_1 &\rightarrow \begin{bmatrix} a \\ b \end{bmatrix} \\ \vec{e}_2 &\rightarrow \begin{bmatrix} -b \\ a \end{bmatrix} \end{aligned} \rightarrow B = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

GOAL:  $\frac{d\vec{x}}{dt} = A\vec{x}$

$$S^{-1}AS = B$$

$$\vec{x}(t) = e^{tA} \vec{x}(0)$$

$$A = SBS^{-1}$$
$$[e^{tA}] = S[e^{tB}]S^{-1}$$

$$= S \underline{[e^{tB}]} S^{-1} \vec{x}(0)$$

$$A = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \quad \lambda I - A = \begin{bmatrix} \lambda & -1 \\ \omega^2 & \lambda \end{bmatrix}$$

$$P_A(\lambda) = \lambda^2 + \omega^2 = 0$$

$$\lambda^2 = -\omega^2$$

$$\lambda = i\omega$$

$$\bar{\lambda} = -i\omega$$

Formally

$$\lambda = i\omega$$

$$\begin{bmatrix} i\omega & -1 \\ \omega^2 & i\omega \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$



$$i\omega\alpha - \beta = 0$$

$$i\omega + \omega\beta = 0$$

$$\rightarrow \beta = i\omega\alpha$$

$$\alpha = 1$$

$$\beta = i\omega$$

Complex eigenvector  $w = \begin{bmatrix} 1 \\ i\omega \end{bmatrix}$

FACT: If  $\vec{w}$  is a complex e-vector assoc with  $\alpha$  e-value  $\lambda = a+ib$

REASON

Then  $\hat{w}$  is a complex e-vector assoc with  $\alpha$  e-value  $\bar{\lambda} = a-ib$

$$\lambda = a+ib \rightarrow \vec{w} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$$\bar{\lambda} = a-ib \rightarrow \hat{w} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$$\begin{bmatrix} A\vec{w} = \lambda\vec{w} \\ A\hat{w} = \bar{\lambda}\hat{w} \end{bmatrix}$$

$$\lambda = i\omega \rightarrow \vec{w} = \begin{bmatrix} 1 \\ i\omega \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ \omega \end{bmatrix} = \vec{u} + i\vec{v}$$

$$\bar{\lambda} = -i\omega \rightarrow \hat{w} = \begin{bmatrix} 1 \\ -i\omega \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - i \begin{bmatrix} 0 \\ \omega \end{bmatrix} = \vec{u} - i\vec{v}$$

Formally  $\{\lambda, \bar{\lambda}\} \rightarrow \{\vec{w}, \vec{w}^{\perp}\}$

$$S = \begin{bmatrix} \vec{w} & \vec{w}^{\perp} \end{bmatrix}, S^{-1}$$

$$S^{-1} A S = D = \begin{bmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{bmatrix}$$

could proceed as in Real e-value case,

$$\text{If } \boxed{\lambda = a + ib} \rightarrow \boxed{\vec{w} = \vec{u} + i\vec{v}}$$

$$A\vec{w} = \lambda\vec{w}$$

$$A\vec{w}^{\perp} = \bar{\lambda}\vec{w}^{\perp}$$

$$A(\vec{u} + i\vec{v}) = (a + ib)(\vec{u} + i\vec{v}) = (a\vec{u} - b\vec{v}) + i(b\vec{u} + a\vec{v})$$

$$A(\vec{u} - i\vec{v}) = (a - ib)(\vec{u} - i\vec{v}) = (a\vec{u} - b\vec{v}) - i(b\vec{u} + a\vec{v})$$

$$A\vec{u} + iA\vec{v} = (a\vec{u} - b\vec{v}) + i(b\vec{u} + a\vec{v})$$

$$A\vec{u} - iA\vec{v} = (a\vec{u} - b\vec{v}) - i(b\vec{u} + a\vec{v})$$

Add:  $\cancel{\lambda} A\vec{u} = \cancel{\lambda} (a\vec{u} - b\vec{v})$   $A\vec{u} = a\vec{u} - b\vec{v}$

Subtracts  $\cancel{\lambda} A\vec{v} = \cancel{\lambda} i(b\vec{u} + a\vec{v})$   $A\vec{v} = b\vec{u} + a\vec{v}$

$$\text{Span} \{ \vec{u}, \vec{v} \} \xrightarrow{A} \text{Span} \{ \vec{u}, \vec{v} \}$$

INVARIANT SUBSPACE  $\{ \vec{v}, \vec{u} \}$

$$\left. \begin{aligned} A\vec{v} &= a\vec{v} + b\vec{u} \\ A\vec{u} &= -b\vec{v} + a\vec{u} \end{aligned} \right\} \left\{ \vec{v}, \vec{u} \right\} \text{ BASIS for } \text{new coords}$$

$$S = \begin{bmatrix} \vec{v} & \vec{u} \end{bmatrix} \quad \begin{aligned} S\vec{e}_1 &= \vec{v} \\ S\vec{e}_2 &= \vec{u} \end{aligned}$$

$$\left\{ \begin{aligned} A S \vec{e}_1 &= a S \vec{e}_1 + b S \vec{e}_2 \\ A S \vec{e}_2 &= -b S \vec{e}_1 + a S \vec{e}_2 \end{aligned} \right\}$$

$$(S^{-1} A S) \vec{e}_1 = a \vec{e}_1 + b \vec{e}_2 = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$(S^{-1} A S) \vec{e}_2 = -b \vec{e}_1 + a \vec{e}_2 = \begin{bmatrix} -b \\ a \end{bmatrix}$$

$$S^{-1} A S = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = B \quad \begin{array}{l} \text{Rotation} \\ \text{Dilation} \end{array}$$

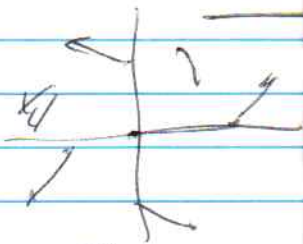
$$A = S B S^{-1}$$

$$[e^{tA}] = S [e^{tB}] S^{-1}$$

$$x(t) = [e^{tA}] \vec{x}(0) = S [e^{tB}] S^{-1} \vec{x}(0)$$

$$\boxed{[e^{tB}] = ?}$$

$$\lambda = a \pm ib$$



Refer to lecture notes for Calculations.

$$\text{Result: } [e^{tB}] = e^{at} \begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix}$$

↑  
exponential growth  
or decay

← time-varying  
ROTATION MATRIX

[a=0 → periodic orbit]

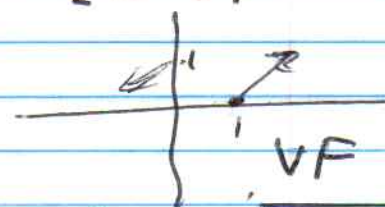
Together ⇒ spiral trajectories

Examples  $\left\{ \begin{array}{l} \frac{dx}{dt} = 2x - 5y \\ \frac{dy}{dt} = 2x - 4y \end{array} \right\}$

$x(0) = 0$

$y(0) = 1$

$$\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix} \quad \frac{d\vec{x}}{dt} = A\vec{x} \quad A = \begin{bmatrix} 2 & -5 \\ 2 & -4 \end{bmatrix}$$

$$\lambda I - A = \begin{bmatrix} \lambda - 2 & 5 \\ -2 & \lambda + 4 \end{bmatrix}$$


$$P_A(\lambda) = \lambda^2 + 2\lambda + 2 = 0$$

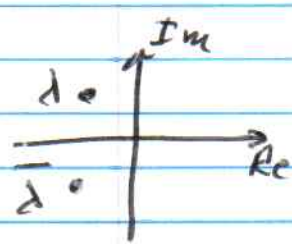
$$\lambda = \frac{-2 \pm \sqrt{4 - 8}}{2}$$

Use  
vector field  
to distinguish  
clockwise vs.  
counterclockwise

$$(\lambda + 1)^2 + 1 = 0$$

$$= -1 \pm i$$

$$\lambda = -1 \pm i$$



$$\lambda = -1 + i$$

$$\bar{\lambda} = -1 - i$$

$$\lambda = -1 + i$$

OK e-values

$$\lambda = -1 + i = a + ib$$

$$a = -1 < 0$$

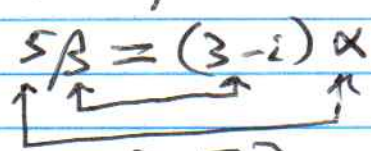
decay

Spiral INWARD

Counterclockwise

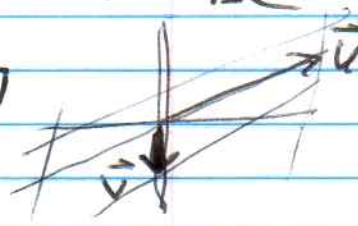
$$\begin{bmatrix} -3+i & 5 \\ -2 & 3+i \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(-3+i)\alpha + 5\beta = 0$$

$$5\beta = (3-i)\alpha$$


$$\vec{w} = \begin{bmatrix} 5 \\ 3-i \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix} + i \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$= \vec{u} + i\vec{v}$$





$$\{\vec{v}, \vec{w}\} = B$$

$$\lambda = \underline{-1 + i} \quad \begin{array}{l} a = -1 \\ b = 1 \end{array}$$

$$S = \begin{bmatrix} 0 & 5 \\ -1 & 3 \end{bmatrix} \quad S^{-1} = \frac{1}{5} \begin{bmatrix} 3 & -5 \\ 1 & 0 \end{bmatrix}$$

$$S^{-1}AS = B = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$$

$$A = SBS^{-1}$$

$$[e^{tA}] = S [e^{tB}] S^{-1}$$

$$[e^{tB}] = e^{-t} \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$$

$$\vec{x}(t) = [e^{tA}] \vec{x}(0) = \begin{bmatrix} 0 & 5 \\ -1 & 3 \end{bmatrix} \frac{e^{-t}}{5} \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} 3 & -5 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{e^{-t}}{5} \begin{bmatrix} 5 \sin t & 5 \cos t \\ -\cos t + 3 \sin t & \sin t + 3 \cos t \end{bmatrix} \begin{bmatrix} -5 \\ 0 \end{bmatrix}$$

$$= -e^{-t} \begin{bmatrix} 5 \sin t & 5 \cos t \\ -\cos t + 3 \sin t & \sin t + 3 \cos t \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= -e^{-t} \begin{bmatrix} 5 \sin t \\ -\cos t + 3 \sin t \end{bmatrix} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

$$\left\{ \begin{array}{l} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -4x + 4y \end{array} \right.$$

Repeated (real)  
eigenvalue case

$$\frac{d\vec{x}}{dt} = A\vec{x} \quad A = \begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix}$$

$$\lambda I - A = \begin{bmatrix} \lambda & -1 \\ 4 & \lambda - 4 \end{bmatrix}$$

$$P(\lambda) = \lambda^2 - 4\lambda + 4 = 0$$

$$= (\lambda - 2)^2 = 0$$

$$\lambda_1 = 2$$

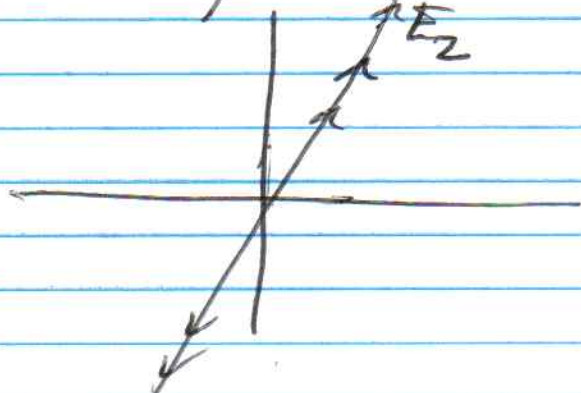
$$AM = 2$$

$$GM = 1$$

$$\lambda_1 = 2 \rightarrow \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$2\alpha - \beta = 0 \Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\beta = 2\alpha$$



only one  
eigenvector

generalized  
e-vector

will be needed

(next lecture)