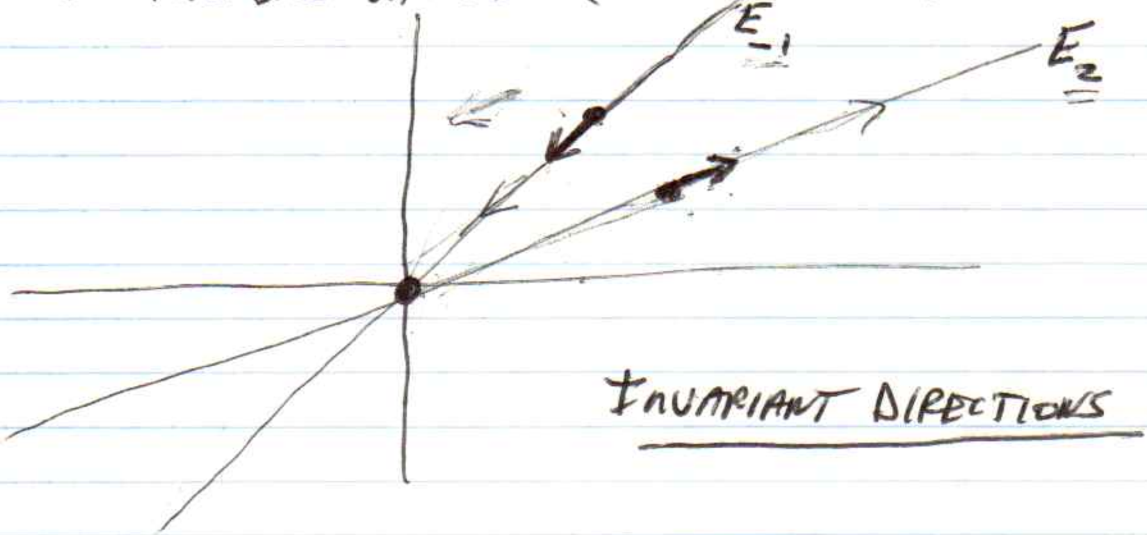
CHARACTERISTIC VECTORS (EIGENVECTOR)
CHARACTERISTIC VALUES (EIGENVALUES)



INVARIANT DIRECTIONS

$\frac{d\vec{x}}{dt} = A\vec{x} \parallel \vec{x} \iff \boxed{A\vec{x} = \lambda\vec{x}}$ λ = lambda

$\frac{d\vec{x}}{dt} = A\vec{x} \quad A = \begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix}$

$\vec{x}$ eigenvector
λ its eigenvalue

Note: $\vec{x}$ an e-vector, then $c\vec{x}$ also an eigenvector

$$A(c\vec{x}) = cA\vec{x} = c\lambda\vec{x} = \lambda(c\vec{x})$$

$c\vec{x}$ e-vctr, same e-value.

Finding e-values + e-vectors.

$$A\vec{x} = \lambda\vec{x} = \lambda I\vec{x} \qquad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A\vec{x} - \lambda I\vec{x} = \vec{0} \rightarrow (A - \lambda I)\vec{x} = \vec{0}$$

$$\lambda I\vec{x} - A\vec{x} = \vec{0} \rightarrow \boxed{(\lambda I - A)\vec{x} = \vec{0}}$$

CONDITION: $A - \lambda I$ or $\lambda I - A$ must not be invertible

FACT: An $n \times n$ MATRIX B is invertible if and only if $\det B \neq 0$.

Condition $\lambda I - A$ not invertible

$$\Leftrightarrow \boxed{\det(\lambda I - A) = 0}$$

Def'n: $p_A(\lambda) = \det(\lambda I - A)$ is called the characteristic polynomial of MATRIX A.

Example: $A = \begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 5-\lambda & -6 \\ 3 & -4-\lambda \end{bmatrix}$$

$$\lambda I - A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} \lambda-5 & 6 \\ -3 & \lambda+4 \end{bmatrix}$$

$$p_A(\lambda) = \det \begin{bmatrix} \lambda-5 & 6 \\ -3 & \lambda+4 \end{bmatrix} = \lambda^2 - \lambda - 20 + 18$$

$$= \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1) = 0$$

$\lambda_1 = 2$ $\lambda_2 = -1$ characteristic VALUES

eigenvalues

Spectrum$\{A\}$ = $\{$eigenvalues$\}$ = $\{2, -1\}$

eigenvalue $\rightarrow$ Eigenvector

$$\lambda I - A = \begin{bmatrix} \lambda-5 & 6 \\ -3 & \lambda+4 \end{bmatrix}$$

$\lambda_1 = 2$ $\quad \begin{bmatrix} -3 & 6 \\ -3 & 6 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} -3\alpha+6\beta \\ -3\alpha+6\beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\vec{v}_1 = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ $\quad -3\alpha + 6\beta = 0$ $\quad \beta = 1$

$\alpha = 2\beta$ $\quad \alpha = 2$

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$\lambda_2 = -1$ $\quad \begin{bmatrix} -6 & 6 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad -6\alpha+6\beta=0$, $\quad -3\alpha+3\beta=0$

$\alpha = \beta$ $\quad \beta = 1 \rightarrow \alpha = 1$ $\quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

NON-MATRIX SOLUTION of

$x(0) = 3$ $\quad \boxed{\frac{d\vec{x}}{dt} = A\vec{x}}$ $\quad \left\{ \begin{aligned} \frac{dx}{dt} &= 5x - 6y \\ \frac{dy}{dt} &= 3x - 4y \end{aligned} \right\}$

$y(0) = 1$

Solution of form $\vec{x}(t) = c_1(t)\,\vec{v}_1 + c_2(t)\,\vec{v}_2$

$$\vec{x}(0) = c_1(0)\,\vec{v}_1 + c_2(0)\,\vec{v}_2$$

$$\frac{d\vec{x}}{dt} = \boxed{\frac{dc_1}{dt}}\,\vec{v}_1 + \boxed{\frac{dc_2}{dt}}\,\vec{v}_2 = A\vec{x}$$

$$= A\left(c_1\vec{v}_1 + c_2\vec{v}_2\right) = c_1 A\vec{v}_1 + c_2 A\vec{v}_2$$

$$= \boxed{c_1\lambda_1}\,\vec{v}_1 + \boxed{c_2\lambda_2}\,\vec{v}_2$$

$$\Rightarrow \boxed{\frac{dc_1}{dt} = \lambda_1 c_1} \quad \boxed{\frac{dc_2}{dt} = \lambda_2 c_2}$$

$$c_1(t) = c_1(0)\, e^{\lambda_1 t} \qquad \lambda_1 = 2 > 0 \text{ growth}$$

$$c_2(t) = c_2(0)\, e^{\lambda_2 t} \qquad \lambda_2 = -1 < 0 \text{ decay.}$$

$$\Rightarrow \vec{x}(t) = \underbrace{c_1(0) e^{\lambda_1 t}}_{\text{growth}} \vec{v}_1 + \underbrace{c_2(0)\, e^{\lambda_2 t}}_{\text{decay}} \vec{v}_2$$

$$= \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} \begin{bmatrix} c_1(0) e^{\lambda_1 t} \\ c_2(0) e^{\lambda_2 t} \end{bmatrix}$$

$$S = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{"change of basis matrix"}$$

$$= \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} c_1(0) \\ c_2(0) \end{bmatrix} = S \begin{bmatrix} e^{tD} \end{bmatrix} \begin{bmatrix} c_1(0) \\ c_2(0) \end{bmatrix}$$

$$\vec{x}(0) = c_1(0)\, \vec{v}_1 + c_2(0)\, \vec{v}_2 = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} \begin{bmatrix} c_1(0) \\ c_2(0) \end{bmatrix}$$

$$\vec{x}(0) = S \begin{bmatrix} c_1(0) \\ c_2(0) \end{bmatrix}$$

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \qquad \begin{bmatrix} e^{tD} \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$$

$$S^{-1} \vec{x}(0) = \begin{bmatrix} c_1(0) \\ c_2(0) \end{bmatrix} \qquad \boxed{\frac{d\vec{x}}{dt} = A\vec{x} \quad x(0)}$$

$$\vec{x}(t) = \boxed{S \begin{bmatrix} e^{tD} \end{bmatrix} S^{-1}}\, \vec{x}(0) = \begin{bmatrix} e^{tA} \end{bmatrix} \vec{x}(0)$$

Solution:

$$\vec{x}(t) = c_1(0)\, e^{\lambda_1 t}\, \vec{v}_1 + c_2(0)\, e^{\lambda_2 t}\, \vec{v}_2$$

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda_1 = 2 \qquad \lambda_2 = -1$$

$$S = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$S \begin{bmatrix} c_1(0) \\ c_2(0) \end{bmatrix} = \vec{x}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1(0) \\ c_2(0) \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$S^{-1} = \frac{1}{\det S} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

$$\therefore \begin{bmatrix} c_1(0) \\ c_2(0) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\vec{x}(t) = 2e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} - e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4e^{2t} - e^{-t} \\ 2e^{2t} - e^{-t} \end{bmatrix} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

CHANGING COORDINATES

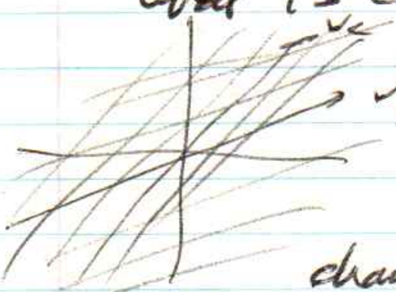
$$\text{MATRIX } A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} \uparrow & \uparrow \\ [A\vec{e}_1]_{\mathcal{E}} & [A\vec{e}_2]_{\mathcal{E}} \\ \downarrow & \downarrow \end{bmatrix}$$

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$A\vec{e}_1 = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} = 2\vec{e}_1 - 1\vec{e}_2$$

$$A\vec{e}_2 = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} = 3\vec{e}_1 + 4\vec{e}_2$$

Q: If, instead of $\mathcal{E} = \{\vec{e}_1, \vec{e}_2\}$, we use $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$ as basis for coordinates, what is corr matrix for this function?



functions $y = f(x) = x^2$

$$\begin{tikzcd} x \arrow[r, "f"] & y \\ u \arrow[u, "g"] \arrow[r, "\tilde{f}"] & z \arrow[u, "h"'] \end{tikzcd}$$

Commuting diagram [See supplement]

changing units

x feet → u inches

y pounds → z ounces

$y = x^2$

$u = 12x \qquad z = 16y = h^{-1}(y) \qquad \frac{z}{16} = \left(\frac{u}{12}\right)^2$

$x = \frac{u}{12} = g(u) \qquad y = \frac{z}{16} = h(z)$

$$z = 16\left(\frac{u}{12}\right)^2 = \tilde{f}(u) = h^{-1} \circ f \circ g(u)$$

$u = g^{-1}(x)$

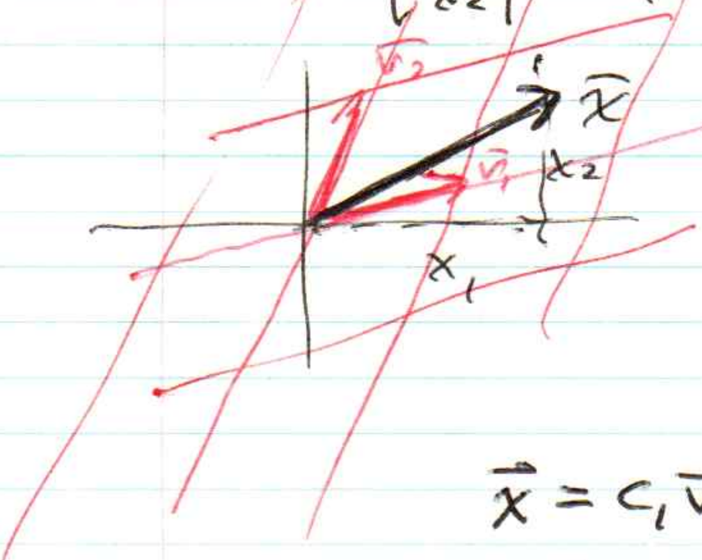
Specifically in case of $n \times n$ MATRIX A

$$A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

and a change of basis $\mathcal{E} \longrightarrow \mathcal{B}$

$$\{\vec{e}_1, \vec{e}_2\} \qquad \{\vec{v}_1, \vec{v}_2\}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \vec{e}_1 + x_2 \vec{e}_2 \quad \text{standard.}$$



$$\vec{x} = c_1 \vec{v}_1 + c_2 \vec{v}_2$$

coords of $\vec{x}$ Relative to BASIS $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$

$$\vec{x} = c_1 \vec{v}_1 + c_2 \vec{v}_2 = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$[\vec{x}]_{\mathcal{E}} = \vec{x} = S[\vec{x}]_{\mathcal{B}}$$

$$S^{-1}\vec{x} = [\vec{x}]_{\mathcal{B}} \quad \text{"contravarient"}$$

MATRIX of LINEAR TRANSFORMATION Relative to a Different BASIS

$$\begin{array}{ccc} \vec{x} \in \{\mathbb{R}^n, \mathcal{E}\} & \xrightarrow{\quad A \quad} & \{\mathbb{R}^n, \mathcal{E}\} \\ \uparrow S & & \uparrow S \\ [\vec{x}]_{\mathcal{B}} \in \{\mathbb{R}^n, \mathcal{B}\} & \xrightarrow[{[A]_{\mathcal{B}}}]{\tilde{A}} & \{\mathbb{R}^n, \mathcal{B}\} \end{array}$$

$$\boxed{[A]_{\mathcal{B}} = S^{-1} A S}$$

$$A = S [A]_{\mathcal{B}} S^{-1}$$

Eigenvector $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\} \equiv \mathcal{B}$

$$S = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ \vdots & & & \end{bmatrix}$$

$$\left\{\begin{array}{l} A\vec{v}_1 = \lambda_1 \vec{v}_1 \\ A\vec{v}_2 = \lambda_2 \vec{v}_2 \\ \quad \vdots \\ A\vec{v}_n = \lambda_n \vec{v}_n \end{array}\right\} \quad \left\{\begin{array}{l} AS\vec{e}_1 = \lambda_1 S\vec{e}_1 \\ \quad \vdots \\ AS\vec{e}_n = \lambda_n S\vec{e}_n \end{array}\right\}$$

$$\left\{\begin{array}{l} S^{-1}AS\,\vec{e}_1 = \lambda_1 \vec{e}_1 \\ \quad \vdots \\ S^{-1}AS\,\vec{e}_n = \lambda_n \vec{e}_n \end{array}\right\} \quad S^{-1}AS = \begin{bmatrix} \lambda_1 & 0 & & 0 \\ 0 & \lambda_2 & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & \lambda_n \end{bmatrix}$$

$$= D$$

$$\boxed{S^{-1}AS = D} = [A]_{\mathcal{B}}$$

Apply to system $\frac{d\vec{x}}{dt} = A\vec{x}$, $\vec{x}(0)$

If A yields a basis of eigenvectors.

(DIAGONALIZABLE), then

$$S = \begin{bmatrix} \vec{v}_1 \cdots \vec{v}_n \end{bmatrix} \qquad S^{-1}AS = D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

$$A = SDS^{-1} \qquad \mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$$

$$\frac{d\vec{x}}{dt} = SDS^{-1}\vec{x}$$

$$S^{-1}\vec{x} = [\vec{x}]_{\mathcal{B}}$$

$$S^{-1}\frac{d\vec{x}}{dt} = DS^{-1}\vec{x}$$

[Lemma: B matrix of constants:
$$\frac{d}{dt}(B\vec{x}) = B\frac{d\vec{x}}{dt}$$]

$$\frac{d}{dt}(S^{-1}\vec{x}) = D(S^{-1}\vec{x}) \Rightarrow \boxed{\frac{d\vec{u}}{dt} = D\vec{u}}$$

$$\vec{u} = S^{-1}\vec{x} = [\vec{x}]_{\mathcal{B}}$$

uncoupled

$$\vec{u}(t) = [e^{tD}]\,\vec{u}(0)$$

FINALLY, change BACK to orig coords

$$S^{-1}\vec{x}(t) = \left[e^{tD}\right] S^{-1}\vec{x}(0)$$

$$\vec{x}(t) = \underbrace{S\left[e^{tD}\right]S^{-1}}_{\left[e^{tA}\right]}\vec{x}(0)$$

$$\vec{x}(t) = \left[e^{tA}\right]\vec{x}(0)$$

$$S^{-1}AS = D$$

$$A = SDS^{-1}$$

$$\left[e^{tA}\right] = S\left[e^{tD}\right]S^{-1}$$

$$\frac{d\vec{x}}{dt} = A\vec{x} \qquad A \to \{\lambda_1, \dots, \lambda_n\}$$

$$\{\vec{v}_1, \dots \vec{v}_n\}$$

$$S = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix} \qquad D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

$$\left[e^{tD}\right] = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix}$$

REDO ORIG PROBLEM

$$\frac{d\vec{x}}{dt} = A\vec{x} \qquad A = \begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix} \qquad \vec{x}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\lambda I - A = \begin{bmatrix} \lambda-5 & 6 \\ -3 & \lambda+4 \end{bmatrix}$$

$$p(\lambda) = \lambda^2 - \lambda - 2 = (\lambda-2)(\lambda+1) = 0$$

$$\lambda_1 = 2 \qquad \lambda_2 = -1$$

$$\downarrow \qquad\qquad \downarrow$$

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$S = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \qquad S^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}.$$

$$x(t) = \left[e^{tA}\right]\vec{x}(0) \qquad S^{-1}AS = D$$

$$A = SDS^{-1}$$

$$\left[e^{tA}\right] = S\left[e^{tD}\right]S^{-1}$$

$$\vec{x}(t) = S\left[e^{tD}\right]S^{-1}\vec{x}(0)$$

$$= \underbrace{\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{bmatrix}}\underbrace{\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix}}$$

$$= \begin{bmatrix} 2e^{2t} & e^{-t} \\ e^{2t} & e^{-t} \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = 2e^{2t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} - e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4e^{2t} - e^{-t} \\ 2e^{2t} - e^{-t} \end{bmatrix} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

SIDE NOTE: CHARACTERISTIC POLYNOMIAL.

$$\boxed{\ddot{x} + 3\dot{x} + 2x = 0}$$ 2nd order ODE.

$x = e^{rt}$

$*\quad p(r) = r^2 + 3r + 2 = (r+2)(r+1) = 0$

$r = -2 \qquad r = -1$

$e^{-2t} \qquad e^{-t}$

$\text{Span}\{e^{-2t}, e^{-t}\} \rightarrow$ all solutions

$$\Rightarrow \boxed{x(t) = C_1 e^{-2t} + C_2 e^{-t}}$$

Reduction of order
$$\left\{ \begin{array}{l} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -2x - 3y \end{array} \right\}$$

$$\frac{d\vec{x}}{dt} = A\vec{x} \qquad A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

Eigenvalues $\quad \lambda I - A = \begin{bmatrix} \lambda & -1 \\ 2 & \lambda+3 \end{bmatrix}$

$*\quad p(\lambda) = \lambda^2 + 3\lambda + 2 = (\lambda+2)(\lambda+1)$

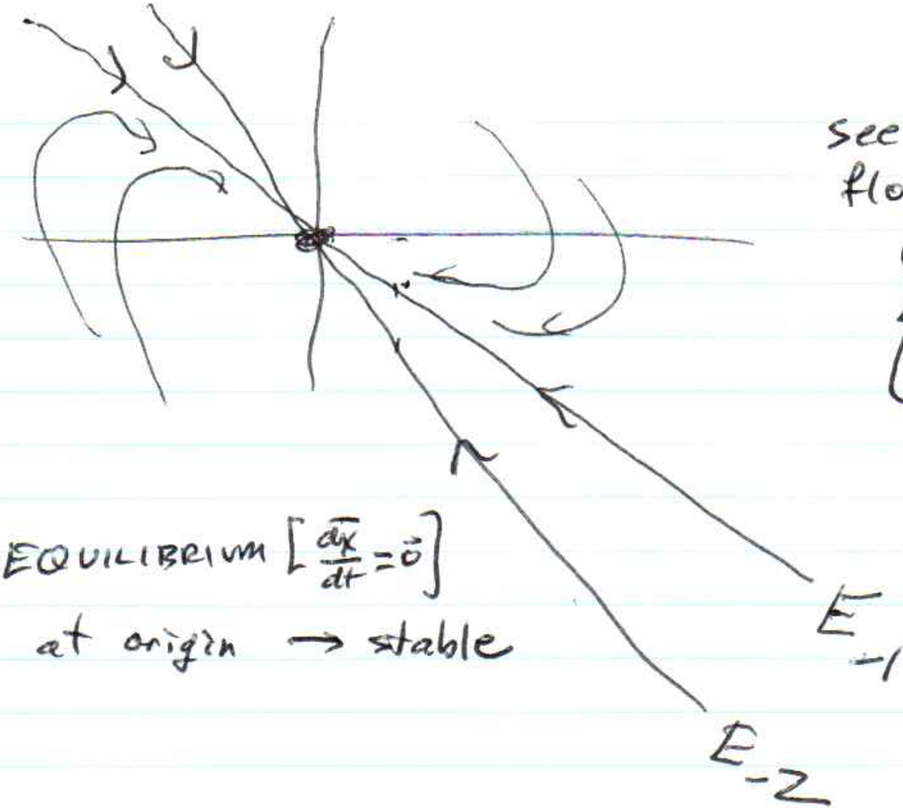
characteristic polynomial is the same

$\lambda_1 = -2 \qquad \lambda_2 = -1$

$-2\alpha - \beta = 0 \qquad -\alpha - \beta = 0$

$\beta = -2\alpha \qquad \beta = -\alpha$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$



See PPLANE for flow

$$\begin{cases} \dfrac{dx}{dt} = y \\[2ex] \dfrac{dy}{dt} = -2x - 3y \end{cases}$$

EQUILIBRIUM $\left[\dfrac{d\vec{x}}{dt} = \vec{0} \right]$

at origin $\rightarrow$ stable