Repeated eigenvalues $\frac{d\vec{x}}{dt} = A\vec{x}$, $\vec{x}(0)$

<u>2nd order ODE:</u> $\ddot{x} + 4\dot{x} + 4x = 0$ $\boxed{\begin{matrix} x(0)=3 \\ \dot{x}(0)=2 \end{matrix}}$

$$x = e^{rt}$$

$$p(r) = r^2 + 4r + 4 = (r+2)^2 = 0$$

$$r = -2 \quad AM = 2$$

$$\mathcal{B} = \{e^{-2t}, te^{-2t}\}$$

$$x(t) = c_1 e^{-2t} + c_2 te^{-2t}$$

✓ ✓ $\Rightarrow$ soln.

<u>Reduction of order</u>

$$\dot{x} = y \qquad \begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -4x - 4y \end{cases} \qquad \begin{matrix} x(0)=3 \\ y(0)=2 \end{matrix}$$

$$\ddot{x} = \dot{y}$$

$$\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix} \quad \frac{d\vec{x}}{dt} = A\vec{x} \quad A = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix}$$

$$\lambda I - A = \begin{bmatrix} \lambda & -1 \\ 4 & \lambda+4 \end{bmatrix}$$

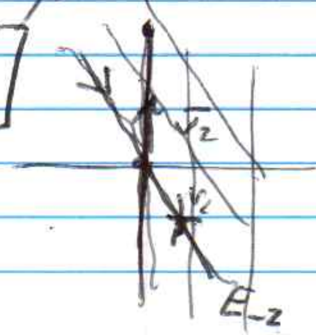
$$p(\lambda) = \lambda^2 + 4\lambda + 4 = (\lambda+2)^2 = 0$$

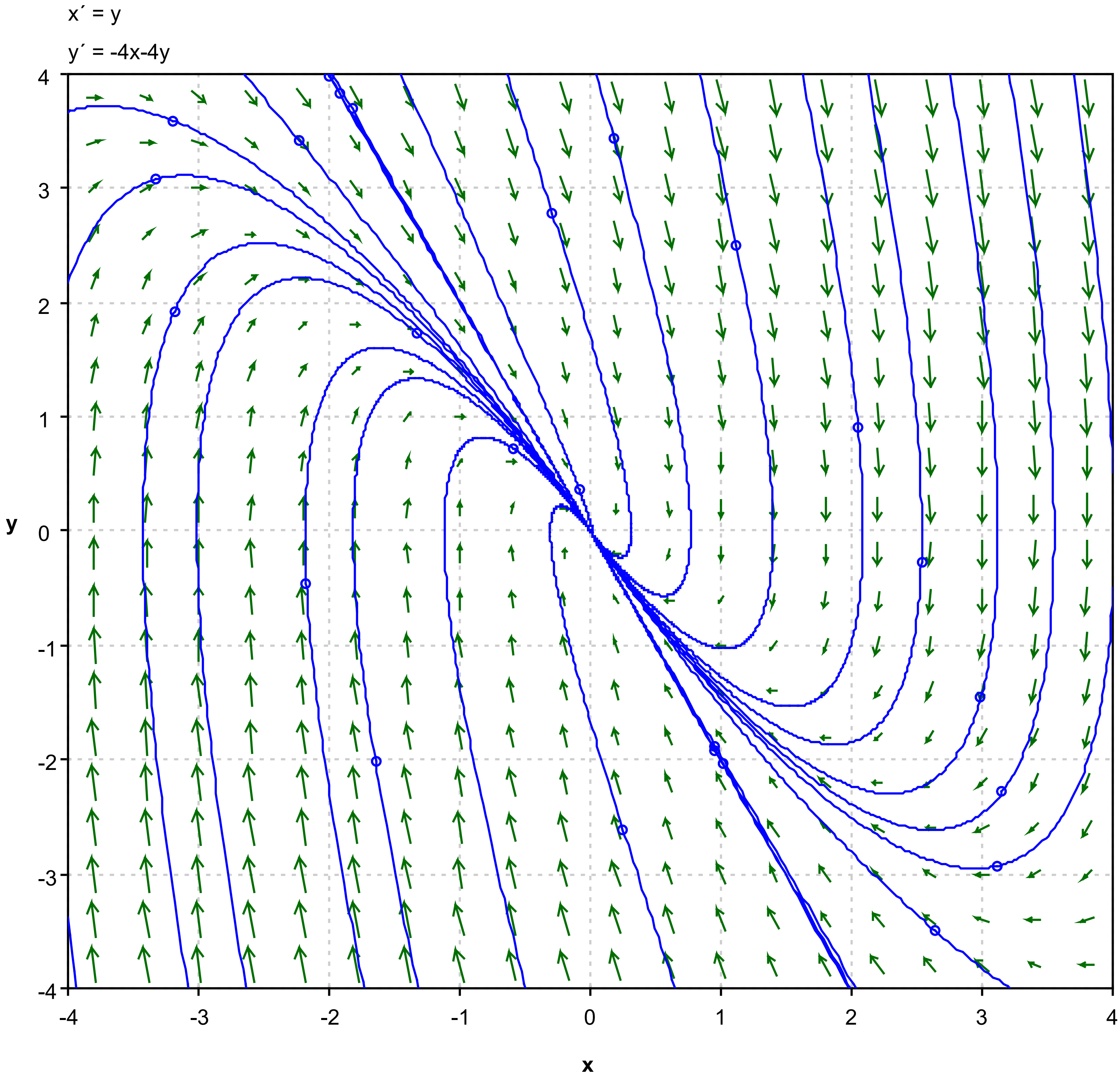
$$\lambda = -2 \quad AM = 2$$

<u>e-vector:</u> $\begin{bmatrix} -2 & -1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad -2\alpha - \beta = 0$

$$\beta = -2\alpha$$

$$\alpha = 1 \rightarrow \beta = -2 \Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$




x´ = y
y´ = -4x-4y
4
3
2
1
0
-1
-2
-3
-4
y
-4
-3
-2
-1
0
1
2
3
4
x

$\{\vec{v}_1, \vec{v}_2\} = \mathcal{B}$ $\quad S^{-1}AS = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}$

$S = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \quad S^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \quad \underset{B}{\shortparallel}$

$S^{-1}AS = B \quad A = SBS^{-1}$

$$\left[e^{tA}\right] = S\left[e^{tB}\right]S^{-1}.$$

$$\vec{x}(t) = \left[e^{tA}\right]\vec{x}(0) = S\left[e^{tB}\right]S^{-1}\vec{x}(0)$$

✓ ↑ ✓ ✓

what is this?

Generally $\begin{cases} \dfrac{du_1}{dt} = \lambda u_1 + u_2 \\ \dfrac{du_2}{dt} = \lambda u_2 \end{cases}$ $\quad B = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$

<u>Q</u>: If $B = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$, what is $\left[e^{tB}\right]$

<u>HW</u>: $\left[e^{tB}\right] = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix} = e^{\lambda t}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$

In example, $\left[e^{tB}\right] = \begin{bmatrix} e^{-2t} & te^{-2t} \\ 0 & e^{-2t} \end{bmatrix} = e^{-2t}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} e^{-2t}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Generalized Eigenvectors

$$\left\{ \begin{aligned} A\vec{v}_1 &= \lambda \vec{v}_1 \\ A\vec{v}_2 &= \vec{v}_1 + \lambda \vec{v}_2 \end{aligned} \right\}$$ seek $\vec{v}_2$ so that these hold, why?

$$\cancel{A\begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} = \lambda \vec{v}_1} \qquad S = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}$$

If so, then:

$$S^{-1}AS = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} = B$$ not diagonal, but in "normal form"

$$(A - \lambda I)\vec{v}_1 = \vec{0} \qquad \vec{v}_1 \in \operatorname{Ker}(A - \lambda I)$$

Actual e-vector

$$(A - \lambda I)\vec{v}_2 = \vec{v}_1$$

$$(A - \lambda I)^2 \vec{v}_2 = (A - \lambda I)\vec{v}_1 = \vec{0}.$$

$$\vec{v}_2 \in \operatorname{Ker}(A - \lambda I)^2$$

Any vector $\vec{v}$ such that

$$(A - \lambda I)^k \vec{v} = \vec{0}$$

is called a <u>generalized e-vector</u>

$$A = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix}$$

$$\lambda I - A = \begin{bmatrix} \lambda & -1 \\ 4 & \lambda + 4 \end{bmatrix}$$

$$(\lambda I - A)\vec{v}_1 = \vec{0}$$

$$(\lambda I - A)\vec{v}_2 = -\vec{v}_1$$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} -2 & -1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$-2\alpha - \beta = -1$$
$$4\alpha + 2\beta = 2$$

$$\alpha = 0 \qquad \beta = 1$$

$$\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

There are other valid choices!

WORST-CASE SCENARIO

$$\frac{d\vec{x}}{dt} = A\vec{x}$$

A 3×3 (or larger)

$$\det(\lambda I - A) = p(\lambda)$$

$$= (\lambda - \lambda_1)^3 = 0 \qquad \lambda = \lambda_1 \quad Am = 3,$$

only one indep e-vector $\vec{v}_1$

$$\left\{\begin{array}{l} A\vec{v}_1 = \lambda\vec{v}_1 \\ A\vec{v}_2 = \vec{v}_1 + \lambda\vec{v}_2 \\ A\vec{v}_3 = \qquad \vec{v}_2 + \lambda\vec{v}_3 \end{array}\right\} \quad \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \mathcal{B}.$$

$$S = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix}$$

$$S^{-1}AS = B = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} \qquad [e^{tB}] = ?$$

$$B = \lambda I + C \qquad C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

See HW for details.

NONLINEAR

$$\frac{d\vec{x}}{dt} = Ax + \vec{b} \qquad \boxed{\frac{d\vec{x}}{dt} - A\vec{x} = \vec{b}} \ *$$

$x_h(t)$ Reminiscent of $\Longrightarrow$ $\dfrac{dx}{dt} - ax = b$

$x_p(t)$ Inhomog, 1st order LINEAR ODE.

$x(t) = x_h(t) + x_p(t)$

Seek $\vec{x}_h(t)$ solves $\dfrac{d\vec{x}}{dt} - A\vec{x} = \vec{0}$ $\left(\dfrac{d\vec{x}}{dt} = A\vec{x}\right)$

Find $\vec{x}_p(t)$ that solves *.

Then $\vec{x}(t) = \overrightarrow{x_h(t)} + \overrightarrow{x_p(t)}$

$\boxed{\text{See notes \#13 for rest of details}}$