NONLINEAR SYSTEMS (2 VARS)

$$\left\{\begin{aligned} \frac{dx}{dt} &= F(x,y,t) \\ \frac{dy}{dt} &= G(x,y,t) \end{aligned}\right\}$$ NONAUTONOMOUS (time-dependent)

$$\left\{\begin{aligned} \frac{dx}{dt} &= F(x,y) \\ \frac{dy}{dt} &= G(x,y) \end{aligned}\right\}$$ Autonomous (time-indep)

POPULATION DYNAMICS : LOTKA-VOLTERRA

$$\left\{\begin{aligned} \frac{1}{x}\frac{dx}{dt} &= a_1 + b_1 x + c_1 y \\ \frac{1}{y}\frac{dy}{dt} &= a_2 + b_2 x + c_2 y \end{aligned}\right\}$$ Relative Growth Rates

$$\left\{\begin{aligned} \frac{dx}{dt} &= (a_1 + b_1 x + c_1 y)x = a_1 x + b_1 x^2 + c_1 xy \\ \frac{dy}{dt} &= (a_2 + b_2 x + c_2 y)y = a_2 y + b_2 xy + c_2 y^2 \end{aligned}\right\}$$

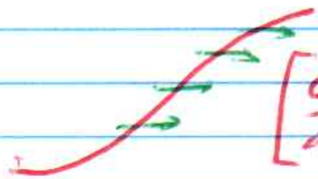
$$\left\{\begin{aligned} \frac{dx}{dt} &= \overset{+}{x}(6 - 2x - y) \\ \frac{dy}{dt} &= \overset{+}{y}(4 - x - y) \end{aligned}\right\}$$ competitive.

NULLCLINE: HORIZONTAL, VERTICAL

$\Rightarrow$ EQUALIBRIA

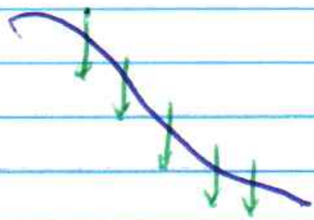
[PHASE-PLANE ANALYSIS]

HORIZONTAL NULLCLINE = curve along which vector field is purely horizontal

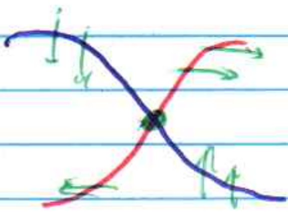


$$\left[\frac{dy}{dt} = G(x,y) = 0\right]$$

VERTICAL NULLCLINE: curve along which vector field is purely vertical.



$$\left[\frac{dx}{dt} = F(x,y) = 0\right]$$



EQUILIBRIUM

$$\frac{dx}{dt} = \frac{dy}{dt} = 0$$

Example $\frac{dx}{dt} = F(x,y) = x(6-2x-y) = 0 \rightarrow$ VNC

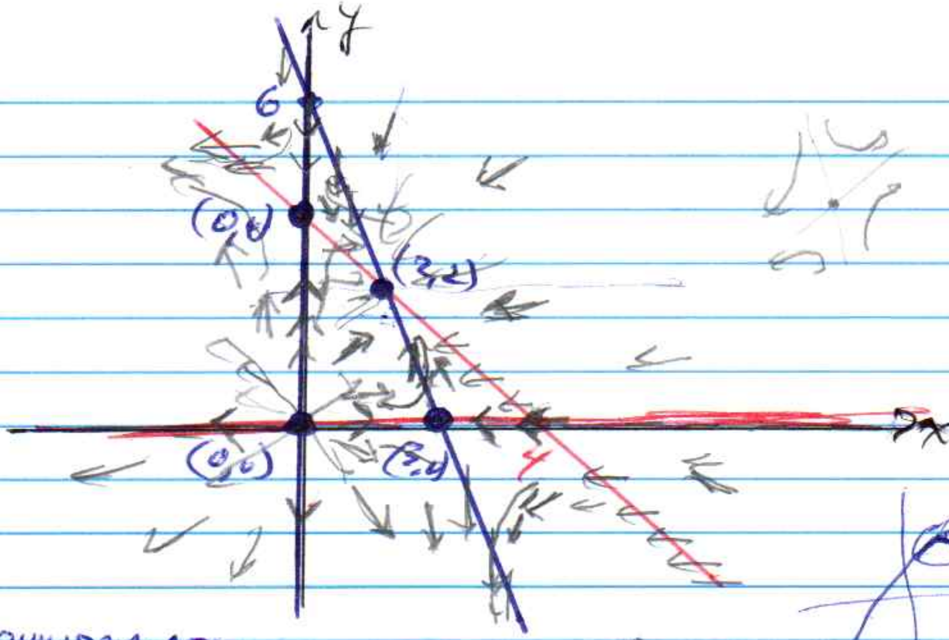
$\frac{dy}{dt} = G(x,y) = y(4-x-y) = 0 \rightarrow$ HNC

VNC: $x(6-2x-y) = 0$ $\quad x = 0 \quad 6-2x-y=0$

$y = -2x+6$

HNC: $y(4-x-y) = 0$ $\quad y=0 \quad 4-x-y=0$

$y = -x+4$



EQUILIBRIA AT:

(0,0)

(3,0)

(0,4)

(2,2)

$y = -2x + 6$

$y = -x + 4$

$-2x + 6 = -x + 4$

$2 = x, \; y = 2$

$(2,2)$

LINEARIZATION NEAR AN EQUILIBRIUM

(x_0, y_0) AN EQUILIBRIUM. $F(x_0, y_0) = 0$

$G(x_0, y_0) = 0$

$$\frac{dx}{dt} = F(x,y) \cong \cancel{F(x_0, y_0)}^{\,0} + F_x(x_0,y_0)(x - x_0) + F_y(x_0,y_0)(y - y_0)$$

$$\frac{dy}{dt} = G(x,y) \cong \cancel{G(x_0, y_0)}^{\,0} + G_x(x_0,y_0)(x - x_0) + G_y(x_0,y_0)(y - y_0)$$

$F_y = \frac{\partial F}{\partial y}$

etc.

$$\boxed{\begin{aligned} \frac{du}{dt} &\cong F_x(\vec{x}_0)\, u + F_y(\vec{x}_0) \checkmark \\ \frac{dv}{dt} &\cong G_x(\vec{x}_0)\, u + G_y(\vec{x}_0) \checkmark \end{aligned}}$$

<u>LINEAR</u>

$\begin{Bmatrix} u = x - x_0 \\ v = y - y_0 \end{Bmatrix}$ $\quad \frac{du}{dt} = \frac{dx}{dt} \qquad \frac{dv}{dt} = \frac{dy}{dt}$

$$\vec{u} = \begin{bmatrix} u \\ v \end{bmatrix} \qquad \frac{d\vec{u}}{dt} = \begin{bmatrix} F_x(\vec{x}_0) & F_y(\vec{x}_0) \\ G_x(\vec{x}_0) & G_y(\vec{x}_0) \end{bmatrix} \vec{u}$$

$$\frac{d\vec{u}}{dt} = \cancel{A}\left[J(\vec{x}_0)\right] \vec{u}$$

Jacobian MATRIX
evaluated at each EQUILIBRIUM

Example:

$$\frac{dy}{dt} = x^2 + y^2 - 8 = F(x,y)$$

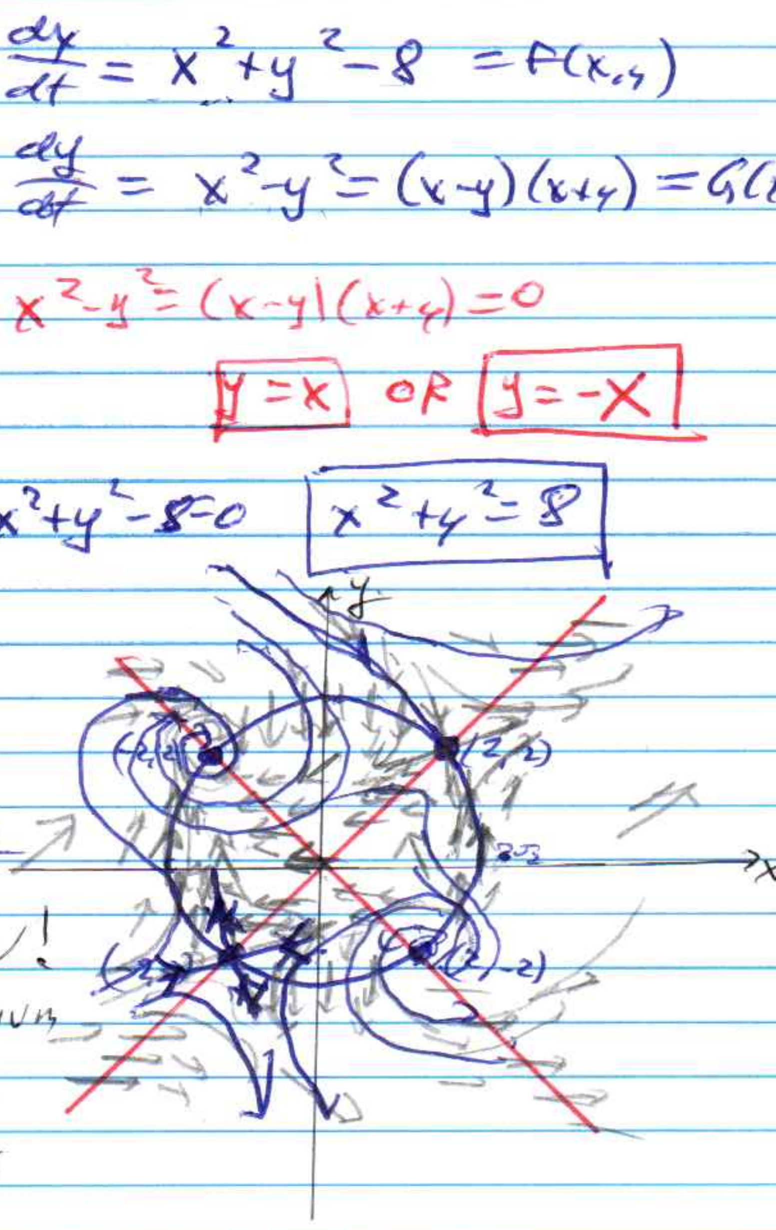
$$\frac{dy}{dt} = x^2 - y^2 = (x-y)(x+y) = G(x,y)$$

HNC: $x^2 - y^2 = (x-y)(x+y) = 0$

$\boxed{y = x}$ OR $\boxed{y = -x}$

VNC: $x^2 + y^2 - 8 = 0$ $\quad \boxed{x^2 + y^2 = 8}$

$x^2 = 4$
$x = 2,\ x = -2$



SPECULATION!
EQUILIBRIUM
Analysis
settles
things

$$\frac{dx}{dt} = x^2 + y^2 - 8 = F(x,y)$$

$$\frac{dy}{dt} = x^2 - y^2 = G(x,y)$$

$$J = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix} = \begin{bmatrix} 2x & 2y \\ 2x & -2y \end{bmatrix}$$

<u>EQ</u>

$(2,2)$

$(2,-2)$

$(-2,2)$

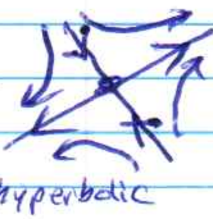
$(-2,-2)$

$$J(2,2) = \begin{bmatrix} 4 & 4 \\ 4 & -4 \end{bmatrix} \qquad p(\lambda) = \lambda^2 - 16 - 16$$

$$= \lambda^2 - 32 = 0$$

$$\lambda I - J = \begin{bmatrix} \lambda-4 & -4 \\ -4 & \lambda+4 \end{bmatrix} \qquad \lambda = \pm 4\sqrt{2}$$

$$+, -$$



hyperbolic

$$J(2,-2) = \begin{bmatrix} 4 & -4 \\ 4 & 4 \end{bmatrix}$$

$$p(\lambda) = (\lambda-4)^2 + 16 = 0$$

$$\lambda I - J = \begin{bmatrix} \lambda-4 & 4 \\ -4 & \lambda-4 \end{bmatrix} \qquad (\lambda-4)^2 = -16$$

$$\lambda - 4 = \pm 4i$$

$$\boxed{\lambda = 4 \pm 4i}$$

$Re(\lambda) > 0$ outward spiral

$$J(-2,2) = \begin{bmatrix} -4 & 4 \\ -4 & -4 \end{bmatrix} \qquad p(\lambda) = (\lambda+4)^2 + 16 = 0$$

$$\lambda = -4 \pm 4i$$

$$\lambda I - J = \begin{bmatrix} \lambda+4 & -4 \\ 4 & \lambda+4 \end{bmatrix}$$

$Re(\lambda) < 0$ inward spiral.

$$J(-2,-2) = \begin{bmatrix} -4 & -4 \\ -4 & 4 \end{bmatrix} \qquad p(\lambda) = \lambda^2 - 32 = 0$$

$$\lambda = \pm 4\sqrt{2}$$

$$+, -$$

$$\lambda I - J = \begin{bmatrix} \lambda+4 & 4 \\ 4 & \lambda-4 \end{bmatrix}$$

hyperbolic