Notes on the dot product and orthogonal projection

An important tool for working with vectors in $\mathbf{R}^n$ (and in abstract vector spaces) is the **dot product** (or, more generally, the inner product). The algebraic definition of the dot product in $\mathbf{R}^n$ is quite simple: Just multiply corresponding components and add.

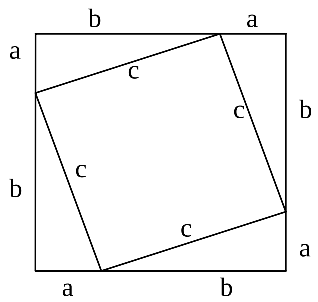
$$\mathbf{u}\cdot\mathbf{v}=\langle u_1,u_2,\cdots,u_n\rangle\cdot\langle v_1,v_2,\cdots,v_n\rangle=u_1v_1+u_2v_2+\cdots+u_nv_n$$

However, the true value of the dot product is realized when you relate this to the measurement of angles using trigonometry and the Law of Cosines.

Here are a couple of classic facts:

I. The <u>Pythagorean Theorem</u>: If a right triangle has legs of length a and b and the hypotenuse has length c, then $a^2+b^2=c^2$.

Proof of the Pythagorean Theorem – Perhaps the easiest way to prove this is with areas:

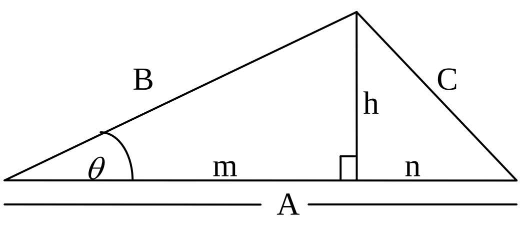


The area of the larger square is the sum of the areas of the smaller square and the four right triangles. This gives us:

$$(a+b)^2=c^2+4\left(\tfrac{1}{2}ab\right)$$
$$a^2+2ab+b^2=c^2+2ab$$
$$a^2+b^2=c^2$$

II. The <u>Law of Cosines</u>: Given any triangle with sides of length A and B adjacent to an angle θ and with the side opposite this angle of length C, then $\mathrm{C}^2=\mathrm{A}^2+\mathrm{B}^2-2\mathrm{AB}\cos\theta$.

Proof of the Law of Cosines – Referring to the variables in the diagram, this is a straightforward application of the Pythagorean Theorem and basic trigonometry. The case for an acute angle shown. The proof is similar for an obtuse angle.



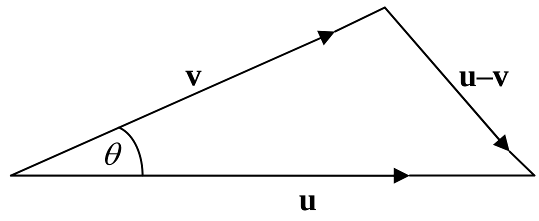
$\mathrm{B}^2=\mathrm{m}^2+\mathrm{h}^2$ and $\mathrm{C}^2=\mathrm{h}^2+\mathrm{n}^2$

$\mathrm{A}=\mathrm{m}+\mathrm{n}$, so we can write $\mathrm{n}=\mathrm{A}-\mathrm{m}$.

If we substitute this, we get:

$$\mathrm{C}^2=\mathrm{h}^2+(\mathrm{A}-\mathrm{m})^2=\mathrm{h}^2+\mathrm{A}^2-2\mathrm{Am}+\mathrm{m}^2$$
$$=\mathrm{B}^2+\mathrm{A}^2-2\mathrm{A}(\mathrm{B}\cos\theta)$$
$$=\mathrm{A}^2+\mathrm{B}^2-2\mathrm{AB}\cos\theta$$

Measuring angles using the dot product: Referring to the “vectorized” diagram to the right, we can restate the Law of Cosines in terms of the lengths of the respective vectors as:



$$\|\mathbf{u}-\mathbf{v}\|^2=\|\mathbf{u}\|^2+\|\mathbf{v}\|^2-2\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta.$$

In order to relate this to the dot product, we need to use a few easy-to-show facts about the dot product, namely:

a) $\mathbf{u}\cdot\mathbf{u}=\|\mathbf{u}\|^2$ where $\|\mathbf{u}\|$ denotes the length of the vector $\mathbf{u}$;

b) $\mathbf{v}\cdot\mathbf{u}=\mathbf{u}\cdot\mathbf{v}$ (commutative law); and

c) $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w})=\mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{w}$ and $(\mathbf{u}+\mathbf{v})\cdot\mathbf{w}=\mathbf{u}\cdot\mathbf{w}+\mathbf{v}\cdot\mathbf{w}$ (left and right distributive laws).

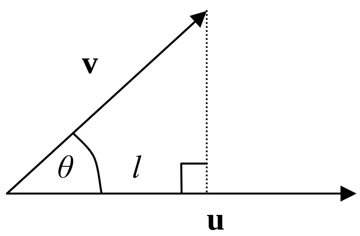
Using these facts, the left-hand side of our vectorized Law of Cosines reads:

$$\|\mathbf{u}-\mathbf{v}\|^2=(\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})=\mathbf{u}\cdot\mathbf{u}-\mathbf{u}\cdot\mathbf{v}-\mathbf{v}\cdot\mathbf{u}+\mathbf{v}\cdot\mathbf{v}=\|\mathbf{u}\|^2+\|\mathbf{v}\|^2-2\mathbf{u}\cdot\mathbf{v}.$$

Comparing this to the original expression, we get the all-important property that $\mathbf{u}\cdot\mathbf{v}=\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta$ where θ is the angle between the two vectors **u** and **v**.

The significance of this property is that the left-hand side is purely algebraic and the right-hand side is purely geometric. This opens the possibility that we can use basic algebraic operations to calculate geometric quantities like lengths and angles. For example, we can rewrite this result as:

$$\cos\theta=\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}.$$



The relation $\mathbf{u}\cdot\mathbf{v}=\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta$ can also be used to provide a simple way of calculating the **<u>scalar projection</u>** of one vector in the direction of another. If we let l denote the orthogonal (perpendicular) projection of **v** in the direction of a another vector **u**, then from the diagram we see that $l=\|\mathbf{v}\|\cos\theta$. We can solve for this in the previous relation to get:

$$l=\|\mathbf{v}\|\cos\theta=\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|}=\mathbf{v}\cdot\frac{\mathbf{u}}{\|\mathbf{u}\|}=\mathbf{v}\cdot(\text{unit vector in the direction of }\mathbf{u}).$$

In other words, if you want to find out "how much" of a vector **v** is in a given direction, you "dot **v** with a unit vector in that direction".

We can further adapt this to find an expression for the **<u>vector projection</u>** of **v** in the direction of **u**. Simply take a unit vector in the direction of **u** and scale it by the scalar projection of **v** in the **u**-direction to construct this vector projection, a vector in the same direction as **u**, but with length equal to the scalar projection. That is:

$$\text{Proj}_{\mathbf{u}}\mathbf{v}=\left(\mathbf{v}\cdot\frac{\mathbf{u}}{\|\mathbf{u}\|}\right)\frac{\mathbf{u}}{\|\mathbf{u}\|}=\left(\frac{\mathbf{v}\cdot\mathbf{u}}{\|\mathbf{u}\|^2}\right)\mathbf{u}.$$

In the special case where the vector **u** is a unit vector, i.e. where $\|\mathbf{u}\|=1$, this simplifies to:

$$\text{Proj}_{\mathbf{u}}\mathbf{v}=(\mathbf{v}\cdot\mathbf{u})\mathbf{u}.$$

The scalar projection of a vector in a given direction is also known as the **component** of the vector in the given direction. It's easy to see that this coincides with the usual x, y, and z components in the case of a vector in $\mathbf{R}^3$. Simply calculate the dot product of the vector $\langle x,y,z\rangle$ with unit vectors in these respective directions. However, with the dot product you can now easily calculate the component of a vector in <u>any</u> direction.

The ability to decompose a vector into its component parts is a fundamental theme in linear algebra. In the case of a more abstract vector space such as a space of functions, this will form the basis of Fourier analysis and other methods for deconstructing functions. These methods play significant roles in fields such as quantum mechanics and digital audio and video recording.