Supplement on systems of linear differential equations – Evolution matrices

Situation: You want to solve a system of first-order linear differential equations of the form $\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$ where **A** is an $n \times n$ real matrix. How is this most efficiently accomplished?

The tool at the heart of these methods is diagonalization or, in the case where a matrix cannot be diagonalized, finding an appropriate change of basis relative to which the underlying linear transformation has the simplest possible matrix representation, i.e. Jordan Canonical Form. A second useful formalism is the use of "evolution matrices."

Suppose **S** is a change of basis matrix corresponding to either diagonalization or reduction to Jordan Canonical Form. We will have $\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{B}$ in this case, where **B** is diagonal or otherwise in simplest form. We then calculate $\mathbf{A} = \mathbf{S}\mathbf{B}\mathbf{S}^{-1}$, and substitution gives $\frac{d\mathbf{x}}{dt} = \mathbf{S}\mathbf{B}\mathbf{S}^{-1}\mathbf{x}$. Multiplying on the left by $\mathbf{S}^{-1}$ and using the basic calculus fact that $\frac{d}{dt}(\mathbf{M}\mathbf{x}) = \mathbf{M}\frac{d\mathbf{x}}{dt}$ for any (constant) matrix **M**, we have $\mathbf{S}^{-1}\frac{d\mathbf{x}}{dt} = \frac{d(\mathbf{S}^{-1}\mathbf{x})}{dt} = \mathbf{B}(\mathbf{S}^{-1}\mathbf{x})$.

If we write $\mathbf{u} = \mathbf{S}^{-1}\mathbf{x} = [\mathbf{x}]_{\mathcal{B}}$, where $\mathcal{B}$ is the new, preferred basis, then in these new coordinates the system becomes $\frac{d\mathbf{u}}{dt} = \mathbf{B}\mathbf{u}$, but now the system will be much more straightforward to solve.

The diagonalizable case

In the case where **B** is a diagonal matrix with the eigenvalues of **A** on the diagonal, the system is just

$$\frac{d\mathbf{u}}{dt} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \mathbf{u} \text{ or } \left\{ \begin{matrix} \frac{du_1}{dt} = \lambda_1 u_1 \\ \vdots \\ \frac{du_n}{dt} = \lambda_n u_n \end{matrix} \right\}.$$

This has the solution $\left\{ \begin{matrix} u_1(t) = e^{\lambda_1 t} u_1(0) \\ \vdots \\ u_n(t) = e^{\lambda_n t} u_n(0) \end{matrix} \right\}$ or $\begin{bmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} u_1(0) \\ \vdots \\ u_n(0) \end{bmatrix}$.

If we use the shorthand notation $[e^{t\mathbf{B}}] = \mathrm{Exp}(t\mathbf{B}) = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix}$, sometimes referred to as the (time-varying) <u>evolution matrix</u> for the simplified system, we can succinctly write the solution as $\mathbf{u}(t) = [e^{t\mathbf{B}}]\mathbf{u}(0)$. To revert back to the original coordinates, we write $\mathbf{x} = \mathbf{S}\mathbf{u}$, so $\mathbf{x}(t) = \mathbf{S}\mathbf{u}(t) = \mathbf{S}[e^{t\mathbf{B}}]\mathbf{u}(0) = \mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1}\mathbf{x}(0)$. If we denote the evolution matrix for the system in its original coordinates as $[e^{t\mathbf{A}}] = Exp(t\mathbf{A})$ where $\mathbf{x}(t) = [e^{t\mathbf{A}}]\mathbf{x}(0)$, then the previous calculation gives the simple relation $[e^{t\mathbf{A}}] = \mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1}$.

In other words, the evolution matrices for the solution are in the same relationship as the matrices **A** and **B**, namely $\mathbf{A} = \mathbf{S}\mathbf{B}\mathbf{S}^{-1}$. This pattern is very easy to remember, and this same pattern will again be the case where **B** is not diagonal but where the corresponding evolution matrix is still relatively easy to calculate.

$\mathbf{A} = \mathbf{S}\mathbf{B}\mathbf{S}^{-1} \quad \Rightarrow \quad [e^{t\mathbf{A}}] = \mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1}$, and the solution of the original system will be $\mathbf{x}(t) = [e^{t\mathbf{A}}]\mathbf{x}(0)$.

The complex eigenvalue case

Suppose we want to solve a system of the form $\frac{d\mathbf{x}}{dt} = \mathbf{Ax}$ where $\mathbf{A}$ is an 2×2 real matrix with a complex conjugate pair of eigenvalues $\lambda = a + ib$ and $\lambda = a - ib$. There are several reasonable ways to proceed, but they all come down to determining the evolution matrix $[e^{t\mathbf{A}}]$ so that we can solve for $\mathbf{x}(t) = [e^{t\mathbf{A}}]\mathbf{x}(0)$.

<u>First, put the system into (real) normal form</u>.

Use the complex eigenvalue $\lambda = a + ib$ to find a complex eigenvector $\mathbf{v} = \mathbf{x} + i\mathbf{y}$. If we change to the basis $\{\mathbf{y}, \mathbf{x}\}$ then, using the change of basis matrix $\mathbf{S} = \begin{bmatrix} \mathbf{y} & \mathbf{x} \end{bmatrix}$, we'll get $\mathbf{S}^{-1}\mathbf{AS} = \mathbf{B} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, a rotation-dilation matrix. Noting, as before, that $\mathbf{A} = \mathbf{SBS}^{-1} \quad \Rightarrow \quad [e^{t\mathbf{A}}] = \mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1}$, we need only to determine $[e^{t\mathbf{B}}]$.

<u>Second, find the evolution matrix for the (real) normal form</u>.

In fact, $[e^{t\mathbf{B}}] = e^{at}\begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix}$, a time-varying rotation matrix with exponential scaling. This yields a trajectory that spirals out in the case where $\operatorname{Re}(\lambda) = a > 0$ (look to the original vector field to see whether it's clockwise or counterclockwise), or a trajectory that spirals inward toward $\mathbf{0}$ in the case where $\operatorname{Re}(\lambda) = a < 0$.

To derive this expression for $[e^{t\mathbf{B}}]$, make another coordinate change with complex eigenvectors starting with $\mathbf{B} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. We know this has the same eigenvalues of $\mathbf{A}$, namely $\lambda = a + ib$ and $\lambda = a - ib$. Use $\lambda = a + ib$ to get the complex eigenvector $\mathbf{w} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$. The eigenvalue $\lambda = a - ib$ will then give eigenvector $\widehat{\mathbf{w}} = \begin{bmatrix} 1 \\ i \end{bmatrix}$. Using the (complex) change of basis matrix $\mathbf{P} = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}$, we have that $\mathbf{P}^{-1}\mathbf{BP} = \mathbf{D} = \begin{bmatrix} a+ib & 0 \\ 0 & a-ib \end{bmatrix}$. It follows that:

$$[e^{t\mathbf{B}}] = \mathbf{P}[e^{t\mathbf{D}}]\mathbf{P}^{-1} = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}\begin{bmatrix} e^{(a+ib)t} & 0 \\ 0 & e^{(a-ib)t} \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} = e^{at}\begin{bmatrix} \frac{e^{ibt}+e^{-ibt}}{2} & -\frac{e^{ibt}-e^{-ibt}}{2i} \\ \frac{e^{ibt}-e^{-ibt}}{2i} & \frac{e^{ibt}+e^{-ibt}}{2} \end{bmatrix} = e^{at}\begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix}.$$

These calculations enable us to write down a closed form expression for the solution of this linear system, namely $\mathbf{x}(t) = [e^{t\mathbf{A}}]\mathbf{x}(0)$ where $[e^{t\mathbf{A}}] = \mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1} = e^{at}\mathbf{S}\begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix}\mathbf{S}^{-1}$. However, the more important result is the ability to qualitatively describe the trajectories for this system by knowing only the real part of the eigenvalues of the matrix $\mathbf{A}$ and the direction of the corresponding vector field (clockwise vs. counterclockwise).

Repeated eigenvalues (with geometric multiplicity less than the algebraic multiplicity)

Suppose we want to solve a system of the form $\frac{d\mathbf{x}}{dt} = \mathbf{Ax}$ where $\mathbf{A}$ is a non-diagonalizable 2×2 real matrix with a repeated eigenvalue λ. We've seen that in this case, we can always find a change of basis matrix $\mathbf{S}$ such that $\mathbf{S}^{-1}\mathbf{AS} = \mathbf{B} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$. As in the previous two cases, $\mathbf{A} = \mathbf{SBS}^{-1} \quad \Rightarrow \quad [e^{t\mathbf{A}}] = \mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1}$ and it comes down to finding $[e^{t\mathbf{B}}]$. This is perhaps most easily done by explicitly solving the corresponding differential equations.

In the new coordinates, this system translates into $\left\{ \begin{aligned} \frac{du_1}{dt} &= \lambda u_1 + u_2 \\ \frac{du_2}{dt} &= \lambda u_2 \end{aligned} \right\}$. The second equation is easily solved to get $u_2(t) = e^{\lambda t} u_2(0)$. We can guess a solution for the first equation of the form $u_1(t) = c_1 t e^{\lambda t} + c_2 e^{\lambda t}$. Differentiating this and substituting into the first equation, we get $c_1(e^{\lambda t} + \lambda t e^{\lambda t}) + c_2 \lambda e^{\lambda t} = \lambda(c_1 t e^{\lambda t} + c_2 e^{\lambda t}) + e^{\lambda t} u_2(0)$. Comparing like terms, we conclude that $c_1 = u_2(0)$. Substituting t = 0, we further conclude that $u_1(0) = c_2$. Putting these results together, we get $u_1(t) = u_2(0) t e^{\lambda t} + u_1(0) e^{\lambda t} = e^{\lambda t} u_1(0) + t e^{\lambda t} u_2(0)$. We therefore have that

$$\mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} e^{\lambda t} u_1(0) + t e^{\lambda t} u_2(0) \\ e^{\lambda t} u_2(0) \end{bmatrix} = \begin{bmatrix} e^{\lambda t} & t e^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix} \begin{bmatrix} u_1(0) \\ u_2(0) \end{bmatrix} = \begin{bmatrix} e^{\lambda t} & t e^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix} \mathbf{u}(0)$$

So, $[e^{t\mathbf{B}}] = \begin{bmatrix} e^{\lambda t} & t e^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix}$ in this case and the solution is given by $\mathbf{x}(t) = \mathbf{S}[e^{t\mathbf{B}}]\mathbf{S}^{-1} = \mathbf{S} \begin{bmatrix} e^{\lambda t} & t e^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix} \mathbf{S}^{-1}\mathbf{x}(0)$.

An alternate method of deriving this result may be found in the homework exercises.

Similar calculations enable us to deal with cases such as a repeated eigenvalue where the geometric multiplicity is 1 and the algebraic multiplicity is 3 (or even worse).

Finally, an actual system may exhibit several of these qualities – one or more complex pairs of eigenvalues, repeated eigenvalues, and distinct real eigenvalues. The Jordan Canonical Form of the matrix for such a system can be analyzed block by block and each of the above solutions applied within each block to determine the evolution matrix for the entire system.

Exercise:

a) Find the general solution for the following system of differential equations:

$$\left\{ \begin{aligned} \frac{dx_1}{dt} &= 2x_1 - 4x_4 + 3x_5 \\ \frac{dx_2}{dt} &= 2x_2 - 2x_3 + 2x_4 \\ \frac{dx_3}{dt} &= x_2 - x_4 \\ \frac{dx_4}{dt} &= -x_4 \\ \frac{dx_5}{dt} &= -3x_4 + 2x_5 \end{aligned} \right\}$$

b) Find the solution in the case where $\mathbf{x}(0)$ = (5, 4, 3, 2, 1).