Math E-21b – Spring 2024 – Homework #9

Problems due online by Sat, April 6, 8:00pm:

Problem 1. Find a 2×2 matrix **A** such that $\mathbf{x}(t)=\begin{bmatrix} 2^t-6^t \\ 2^t+6^t \end{bmatrix}$ is a trajectory of the dynamical system $\mathbf{x}(t+1)=\mathbf{A}\mathbf{x}(t)$. [**Note**: This also translates into $\mathbf{x}(t)=\mathbf{A}^t\mathbf{x}(0)$ for some initial $\mathbf{x}(0)$.]

Problem 2. Find a 2×2 matrix **A** such that $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ are eigenvectors of **A**, with eigenvalues 5 and 10, respectively.

For each of the matrices in Problems 3-7, find all real eigenvalues, their algebraic multiplicities, their geometric multiplicities, and their corresponding eigenvectors. If the matrix is diagonalizable, find an eigenbasis. Show all your work. Do not use technology.

Problem 3. $\mathbf{A}=\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$ **Problem 4.** $\mathbf{A}=\begin{bmatrix} 0 & 4 \\ -1 & 4 \end{bmatrix}$ **Problem 5.** $\mathbf{A}=\begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix}$

Problem 6. $\mathbf{A}=\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ **Problem 7.** $\mathbf{A}=\begin{bmatrix} 1 & 0 & 0 \\ -5 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

Problem 8. If $p(t)=a_nt^n+a_{n-1}t^{n-1}+\cdots+a_1t+a_0$ is a polynomial, and if **A** is a square ($m\times m$) matrix, we define $p(\mathbf{A})=a_n\mathbf{A}^n+a_{n-1}\mathbf{A}^{n-1}+\cdots+a_1\mathbf{A}+a_0\mathbf{I}$. Suppose **v** an eigenvector of the $m\times m$ matrix **A**, with eigenvalue λ. Explain why **v** is an eigenvector of $p(\mathbf{A})$. What is its associated eigenvalue?

Problem 9. Suppose matrix **A** is similar to **B**. What is the relationship between the characteristic polynomials of **A** and **B**? What does your answer tell you about the eigenvalues of **A** and **B** and their algebraic multiplicities?

Problem 10. Consider an arbitrary $n\times n$ matrix **A**. What is the relationship between the characteristic polynomials of **A** and its transpose $\mathbf{A}^{\mathrm{T}}$? What does your answer tell you about the eigenvalues of **A** and $\mathbf{A}^{\mathrm{T}}$?

Problem 11. Two interacting populations of hares and foxes can be modeled by the recursive equations:

$$\begin{aligned} h(t+1) &= 4h(t)-2f(t) \\ f(t+1) &= h(t)+f(t) \end{aligned}$$

For each of the initial populations given in parts (a) through (c), find closed formulas for $h(t)$ and $f(t)$.

a. $h(0)=f(0)=100$ b. $h(0)=200, f(0)=100$ c. $h(0)=600, f(0)=500$

Problem 12. If a 2×2 matrix **A** has two eigenvalues λ_1 and λ_2, show that $\mathrm{tr}(\mathbf{A})=\lambda_1+\lambda_2$.

Problem 13. Prove that if an $n\times n$ matrix **A** has *n* eigenvalues $\lambda_1,\ldots,\lambda_n$, listed with their algebraic multiplicities, then $\mathrm{tr}(\mathbf{A})=\lambda_1+\cdots+\lambda_n$.

Problem 14. Consider all 2×2 matrices **A** of the form $\mathbf{A}=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where *a*, *b*, *c*, and *d* are positive numbers such that $a+c=b+d=1$. Such a matrix is called a *regular transition matrix*. Verify that $\begin{bmatrix} b \\ c \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ are eigenvectors of **A**. What are the associated eigenvalues? Is the absolute value of these eigenvalues more or less than 1? Sketch a phase portrait.

Problem 15. a. Find all eigenvalues and respective eigenvectors of the matrix $\mathbf{A}=\begin{bmatrix}0.5 & 0.25\\0.5 & 0.75\end{bmatrix}$.

b. Sketch a <u>phase portrait</u> of the dynamical system $\mathbf{x}(t+1)=\begin{bmatrix}0.5 & 0.25\\0.5 & 0.75\end{bmatrix}\mathbf{x}(t)$.

c. Find closed formulas for the components of the dynamical system $\mathbf{x}(t+1)=\begin{bmatrix}0.5 & 0.25\\0.5 & 0.75\end{bmatrix}\mathbf{x}(t)$, with initial value $\mathbf{x}_0=\mathbf{e}_1$. Then do the same for the initial value $\mathbf{x}_0=\mathbf{e}_2$. Sketch the two trajectories.

d. For the matrix $\mathbf{A}=\begin{bmatrix}0.5 & 0.25\\0.5 & 0.75\end{bmatrix}$, using technology, compute some powers of the matrix **A**, say, $\mathbf{A}^2,\mathbf{A}^5,\mathbf{A}^{10},\ldots$. What do you observe? Explain your answer carefully.

e. If $\mathbf{A}=\begin{bmatrix}a & b\\c & d\end{bmatrix}$ is an arbitrary regular transition matrix, what can you say about the powers $\mathbf{A}^t$ as *t* goes to infinity?

Problem 16. Some years ago, the Broadway Marketplace opened a couple of blocks from Harvard, selling fresh meat and produce and some groceries and maintaining a constant weekly customer base of 2,000 people. For argument's sake, let's say that a new store, Megamart Inc., opens nearby. Weekly surveys show that Broadway keeps 50% of its customers from one week to the next, with the rest going for the cheaper prices at the Megamart, and that Megamart retains 70% of its customers from the previous week with the rest going to Broadway. The state of grocery shopping can be represented by the vector $\mathbf{x}(t)=\begin{bmatrix}B(t)\\M(t)\end{bmatrix}$, where $B(t)$ and $M(t)$ are the numbers of customers shopping at Broadway Marketplace and Megamart, respectively, *t* weeks after Megamart opened. Initially, $B(0)=2000$ and $M(0)=0$.

a. Find a matrix **A** such that $\mathbf{x}(t+1)=\mathbf{A}\mathbf{x}(t)$. Verify that **A** is a regular transition matrix. [See above.]

b. How many customers will shop at each store after *t* weeks? Give closed formulas.

c. The Broadway Marketplace fears that they must close down when they have less than 700 customers per week. Does that happen and, if so, after how many weeks will this happen?

For additional practice:
Section 7.1:
In Exercises 1 through 4, let **A** be an invertible $n\times n$ matrix and **v** an eigenvector of **A** with associated eigenvalue λ.

1. Is **v** an eigenvector of $\mathbf{A}^3$? If so, what is the eigenvalue?
2. Is **v** an eigenvector of $\mathbf{A}^{-1}$? If so, what is the eigenvalue?
3. Is **v** an eigenvector of $\mathbf{A}+2\mathbf{I}_n$? If so, what is the eigenvalue?
4. Is **v** an eigenvector of 7**A**? If so, what is the eigenvalue?
5. If a vector **v** is an eigenvector of both **A** and **B**, is **v** necessarily an eigenvector of $\mathbf{A}+\mathbf{B}$?
6. If a vector **v** is an eigenvector of both **A** and **B**, is **v** necessarily an eigenvector of $\mathbf{AB}$?

Arguing geometrically, find all eigenvectors and eigenvalues of the linear transformations in Exercises 15 through 21. Find a basis consisting of eigenvectors if possible.

15. Reflection about a line L in $\mathbf{R}^2$.
16. Rotation through an angle of 180° in $\mathbf{R}^2$.
17. Counterclockwise rotation through an angle of 45° followed by a scaling by 2 in $\mathbf{R}^2$.
18. Reflection about a plane V in $\mathbf{R}^3$.
19. Orthogonal projection onto a line L in $\mathbf{R}^3$.
20. Rotation about the $\mathbf{e}_3$-axis through an angle of 90°, counterclockwise as viewed from the positive $\mathbf{e}_3$-axis in $\mathbf{R}^3$.
21. Scaling by 5 in $\mathbf{R}^3$.

39. Find a basis of the linear space V of 2×2 matrices $\mathbf{A}$ for which $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is an eigenvector, and thus determine the dimension of V.

53. Three holy men (let's call them Abraham, Benjamin, and Chaim) put little stock in material things; their only earthly possession is a small purse with a bit of gold dust. Each day they get together for the following bizarre bonding ritual: Each of them takes his purse and gives his gold away to the two others, in equal parts. For example, if Abraham has 4 ounces one day, he will give 2 ounces each to Benjamin and Chaim.

    a. If Abraham starts out with 6 ounces, Benjamin with 1 ounce, and Chaim with 2 ounces, find formulas for the amounts $a(t)$, $b(t)$, and $c(t)$ each will have after t distributions.

    Hint: The vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ will be useful.

    b. Who will have the most gold after one year, that is, after 365 distributions?

Section 7.2:

For the matrices in Exercises 5 and 7, find all real eigenvalues, with their algebraic multiplicities. Show all your work. Do not use technology.

5. $\begin{bmatrix} 11 & -15 \\ 6 & -7 \end{bmatrix}$ 7. $\mathbf{I}_3$

15. Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & k \\ 1 & 1 \end{bmatrix}$, where k is an arbitrary constant. For which values of k does $\mathbf{A}$ have two distinct real eigenvalues? When is there no real eigenvalue?

Section 7.3:

For the matrices in Exercises 11 and 16, find all (real) eigenvalues. Then find a basis for each eigenspace, and find an eigenbasis, if you can. Do not use technology.

11. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ 16. $\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 2 & 2 & 0 \end{bmatrix}$

21. Find a 2×2 matrices $\mathbf{A}$ for which $E_1 = \text{span}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $E_2 = \text{span}\begin{bmatrix} 2 \\ 3 \end{bmatrix}$. How many such matrices are there?