Math E-21b – Spring 2025 – Homework #8

Problem 1. (6.1/18) Use the determinant to find out for which values of the constant k the matrix $\begin{bmatrix} 0 & 1 & k \\ 3 & 2k & 5 \\ 9 & 7 & 5 \end{bmatrix}$ is invertible.

In Problems 2 and 3, use the determinant to find out for which values of the constant λ the matrix $\mathbf{A}-\lambda\mathbf{I}_n$ fails to be invertible.

Problem 2. (6.1/26) $\mathbf{A}=\begin{bmatrix} 4 & 2 \\ 2 & 7 \end{bmatrix}$ **Problem 3.** (6.1/30) $\mathbf{A}=\begin{bmatrix} 4 & 2 & 0 \\ 4 & 6 & 0 \\ 5 & 2 & 3 \end{bmatrix}$

Problem 4. (6.1/34) Find the determinant of the matrix $\begin{bmatrix} 4 & 5 & 0 & 0 \\ 3 & 6 & 0 & 0 \\ 2 & 7 & 1 & 4 \\ 1 & 8 & 2 & 3 \end{bmatrix}$. [**Do it <u>without</u> the calculator.**]

Problem 5. (6.1/44) If **A** is an $n \times n$ matrix and k is an arbitrary constant, what is the relationship between det(**A**) and det(k**A**)?

Problem 6. Use Gaussian elimination, i.e. row reduction, to find the determinant of the matrix:

(a) (6.2/6) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 4 & 4 \\ 1 & -1 & 2 & -2 \\ 1 & -1 & 8 & -8 \end{bmatrix}$ (b) (6.2/9) $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 \\ 1 & 1 & 3 & 3 & 3 \\ 1 & 1 & 1 & 4 & 4 \\ 1 & 1 & 1 & 1 & 5 \end{bmatrix}$

Problem 7. (6.2/17) Find the determinant of the linear transformation $T(f)=2f+3f'$ from P_2 to P_2.

Problem 8. (6.2/18) Find the determinant of the linear transformation $(T(f))(t)=f(3t-2)$ from P_2 to P_2.

Problem 9. (6.2/25) Find the determinant of the linear transformation $T(\mathbf{M})=\begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}\mathbf{M}$ from the space V of 2×2 upper triangular matrices to V.

Problem 10. (6.2/26) Find the determinant of the linear transformation $T(\mathbf{M})=\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\mathbf{M}+\mathbf{M}\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ from the space V of symmetric 2×2 matrices to V.

Problem 11. (6.2/34) a. For an invertible $n\times n$ matrix **A** and an arbitrary $n\times n$ matrix **B**, show that $\text{rref}[\mathbf{A}\,|\,\mathbf{AB}]=[\mathbf{I}_n\,|\,\mathbf{B}]$.

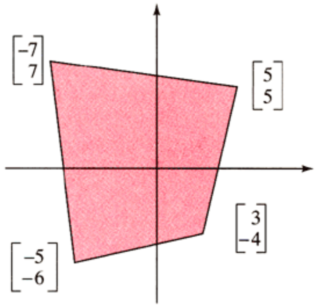
Hint: The left part of $\text{rref}[\mathbf{A}\,|\,\mathbf{AB}]$ is $\text{rref}(\mathbf{A})=\mathbf{I}_n$. Write $\text{rref}[\mathbf{A}\,|\,\mathbf{AB}]=[\mathbf{I}_n\,|\,\mathbf{M}]$; we have to show that $\mathbf{M}=\mathbf{B}$. To demonstrate this, note that the columns of matrix $\begin{bmatrix} \mathbf{B} \\ -\mathbf{I}_n \end{bmatrix}$ are in the kernel of $[\mathbf{A}\,|\,\mathbf{AB}]$ and therefore in the kernel of $[\mathbf{I}_n\,|\,\mathbf{M}]$.

b. What does the formula $\text{rref}[\mathbf{A}\,|\,\mathbf{AB}]=[\mathbf{I}_n\,|\,\mathbf{B}]$ tell you if $\mathbf{B}=\mathbf{A}^{-1}$?

[<u>**Note**</u>: The result in part (a) is used in proving that det(**AB**) = det(**A**)det(**B**) for $n\times n$ matrices **A** and **B**.]

Problem 12. (6.2/40) If **A** is an orthogonal matrix, what are the possible values of det(**A**)?

Problem 13. (6.3/7) Find the area of the following region:



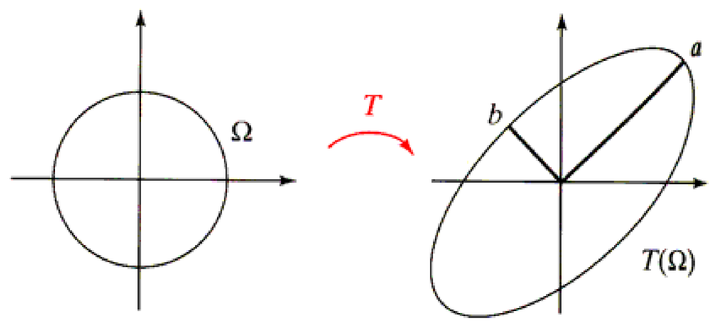
Problem 14. (6.3/13) Find the area (or 2-volume) of the parallelogram (or 2-parallelepiped) defined by the vectors

$$\begin{bmatrix}1\\1\\1\\1\end{bmatrix} \text{ and } \begin{bmatrix}1\\2\\3\\4\end{bmatrix}.$$

Problem 15. (6.3/14) Find the 3-volume of the 3-parallepipied defined by the vectors

$$\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\\4\end{bmatrix}.$$

Problem 16. (6.3/18) If $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$ is an invertible linear transformation from $\mathbf{R}^2$ to $\mathbf{R}^2$, then the image $T(\Omega)$ of the unit circle Ω is an ellipse. (See Exercise 2.2.50.)



a. Sketch this ellipse when $\mathbf{A} = \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix}$, where p and q are positive. What is its area?

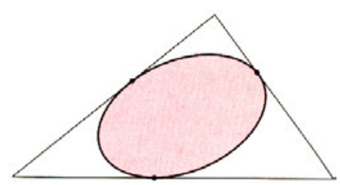
b. For an arbitrary invertible transformation $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$, denote the lengths of the semi-major and semi-minor axes of $T(\Omega)$ by a and b, respectively. What is the relationship between a, b, and det(**A**)?

c. For the transformation $T(\mathbf{x}) = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}\mathbf{x}$, sketch this ellipse and determine its axes.

Hint: Consider $T\begin{bmatrix}1\\1\end{bmatrix}$ and $T\begin{bmatrix}1\\-1\end{bmatrix}$.

Problem 17. (6.3/24) Use Cramer's rule to solve the system $\left\{\begin{matrix} 2x+3y & & = 8 \\ & 4y+5z & = 3 \\ 6x & +7z & = -1 \end{matrix}\right\}$.

Problem 18. (6.3/48) What is the area of the largest ellipse you can inscribe into a triangle with side lengths 3, 4, and 5. *Hint*: The largest ellipse you can inscribe into an equilateral triangle is a circle.



For additional practice:
Section 6.1:

9. Find the determinant of the matrix $\begin{bmatrix} 0 & 1 & 2 \\ 7 & 8 & 3 \\ 6 & 5 & 4 \end{bmatrix}$ and determine if this matrix is invertible.

In Exercises 16 and 17, use the determinant to find out for which values of the constant k the matrix is invertible.

16. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & k & 5 \\ 6 & 7 & 8 \end{bmatrix}$ 17. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & k & -1 \\ 1 & k^2 & 1 \end{bmatrix}$

43. If **A** is an $n \times n$ matrix and k is an arbitrary constant, what is the relationship between det(**A**) and det(–**A**)?

Section 6.2:

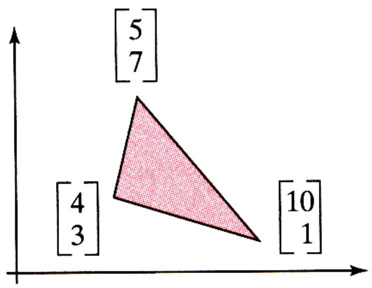
5. Use Gaussian elimination, i.e. row reduction, to find the determinant of the matrix $\begin{bmatrix} 0 & 2 & 3 & 4 \\ 0 & 0 & 0 & 4 \\ 1 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \end{bmatrix}$.

41. Consider a skew-symmetric $n \times n$ matrix **A**, where *n* is odd. Show that **A** is noninvertible, by showing that det(**A**) = 0.

43. Consider two vectors **v** and **w** in $\mathbf{R}^n$. Form the matrix $\mathbf{A} = \begin{bmatrix} \mathbf{v} & \mathbf{w} \end{bmatrix}$.Express det($\mathbf{A}^T\mathbf{A}$) in terms of $\|\mathbf{v}\|$, $\|\mathbf{w}\|$, and $\mathbf{v} \cdot \mathbf{w}$. What can you say about the sign of the result?

Section 6.3:

3. Find the area of the following triangle:



19. A basis $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ of $\mathbf{R}^3$ is called positively oriented if $\mathbf{v}_1$ encloses an acute angle with $\mathbf{v}_2 \times \mathbf{v}_3$. Illustrate this definition with a sketch. Show that the basis is positively oriented if (and only if) $\det\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}$ is positive.

20. We say that a linear transformation *T* from $\mathbf{R}^3$ to $\mathbf{R}^3$ *preserves orientation* if it transforms any positively oriented basis into another positively oriented basis. (See Exercise 19.) Explain why a linear transformation $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$ preserves orientation if (and only if) det(**A**) is positive.

23. Use Cramer's rule to solve the system $\left\{ \begin{array}{l} 5x_1 - 3x_2 = 1 \\ -6x_1 + 7x_2 = 0 \end{array} \right\}$.

Chapter 6 True/False Exercises

1. If $A = \begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix}$ is any 3×3 matrix, then $\det A = \vec{u} \cdot (\vec{v} \times \vec{w})$.

2. $\det(4A) = 4 \det A$ for all 4×4 matrices A.

3. $\det(A + B) = \det A + \det B$ for all 5×5 matrices A and B.

4. The equation $\det(-A) = \det A$ holds for all 6×6 matrices.

5. If all the entries of a 7×7 matrix A are 7, then $\det A$ must be 7^7.

6. An 8×8 matrix fails to be invertible if (and only if) its determinant is nonzero.

7. If B is obtained be multiplying a column of A by 9, then the equation $\det B = 9 \det A$ must hold.

8. $\det(A^{10}) = (\det A)^{10}$ for all 10×10 matrices A.

9. The determinant of any diagonal $n \times n$ matrix is the product of its diagonal entries.

10. If matrix B is obtained by swapping two rows of an $n \times n$ matrix A, then the equation $\det B = -\det A$ must hold.

11. Matrix $\begin{bmatrix} 9 & 100 & 3 & 7 \\ 5 & 4 & 100 & 8 \\ 100 & 9 & 8 & 7 \\ 6 & 5 & 4 & 100 \end{bmatrix}$ is invertible.

12. If A is an invertible $n \times n$ matrix, then $\det(A^T)$ must equal $\det(A^{-1})$.

13. If the determinant of a 4×4 matrix A is 4, then its rank must be 4.

14. There exists a nonzero 4×4 matrix A such that $\det A = \det(4A)$.

15. If two $n \times n$ matrices A and B are similar, then the equation $\det A = \det B$ must hold.

16. The determinant of all orthogonal matrices is 1.

17. If A is any $n \times n$ matrix, then $\det(AA^T) = \det(A^T A)$.

18. There exists an invertible matrix of the form $\begin{bmatrix} a & e & f & j \\ b & 0 & g & 0 \\ c & 0 & h & 0 \\ d & 0 & i & 0 \end{bmatrix}$.

19. The matrix $\begin{bmatrix} k^2 & 1 & 4 \\ k & -1 & -2 \\ 1 & 1 & 1 \end{bmatrix}$ is invertible for all positive constants k.

20. $\det \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} = 1$.

21. There exists a 4×4 matrix A whose entries are all 1 or -1, and such that $\det A = 16$.

22. If the determinant of a 2×2 matrix A is 4, then the inequality $\|A\vec{v}\| \leq 4\|\vec{v}\|$ must hold for all vectors $\vec{v}$ in $\mathbb{R}^2$.

23. If $A = \begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix}$ is a 3×3 matrix, then the formula $\det(A) = \vec{v} \cdot (\vec{u} \times \vec{w})$ must hold.

24. There exist invertible 2×2 matrices A and B such that $\det(A + B) = \det A + \det B$.

25. If all the entries of a square matrix are 1 or 0, then $\det A$ must be 1, 0, or -1.

26. If all the entries of a square matrix A are integers and $\det A = 1$, then the entries of matrix A^{-1} must be integers as well.

27. If all the columns of a square matrix A are unit vectors, then the determinant of A must be less than or equal to 1.

28. If A is any noninvertible square matrix, then $\det A = \det(\text{rref } A)$.

29. If the determinant of a square matrix is -1, then A must be an orthogonal matrix.

30. If all the entries of an invertible matrix A are integers, then the entries of A^{-1} must be integers as well.

31. There exist invertible 3×3 matrices A and S such that $S^{-1}AS = 2A$.

32. There exist invertible 3×3 matrices A and S such that $S^T AS = -A$.

33. If A is any symmetric matrix, then $\det A = 1$ or $\det A = -1$.

34. If A is any skew-symmetric 4×4 matrix, then $\det A = 0$.

35. If $\det A = \det B$ for two $n \times n$ matrices A and B, then A must be similar to B.

36. Suppose A is an $n \times n$ matrix and B is obtained from A by swapping two rows of A. If $\det B < \det A$, then A must be invertible.

37. If an $n \times n$ matrix A is invertible, then there must be an $(n-1) \times (n-1)$ submatrix of A (obtained by deleting a row and a column of A) that is invertible as well.

38. If all the entries of matrices A and A^{-1} are integers, then the equation $\det A = \det(A^{-1})$ must hold.

39. If a square matrix A is invertible, then its classical adjoint $\text{adj}(A)$ is invertible as well.

40. There exists a 3×3 matrix A such that $A^2 = -I_3$.

41. If all the diagonal entries of an $n \times n$ matrix A are odd integers and all the other entries are even integers, then A must be an invertible matrix.

42. If all the diagonal entries of an $n \times n$ matrix A are even integers and all the other entries are odd integers, then A must be an invertible matrix.

43. For every nonzero 2×2 matrix A there exists a 2×2 matrix B such that $\det(A + B) \neq \det A + \det B$.

44. If A is a 4×4 matrix whose entries are all 1 or -1, then $\det A$ must be divisible by 8 [i.e., $\det A = 8k$ for some integer k].

45. If A is an invertible $n \times n$ matrix, then A must commute with its adjoint, $\text{adj}(A)$.

46. There exists a real number k such that the matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & k & 7 \\ 8 & 9 & 8 & 7 \\ 0 & 0 & 6 & 5 \end{bmatrix}$$

is invertible.

47. If A and B are orthogonal $n \times n$ matrices such that $\det A = \det B = 1$, then matrices A and B must commute.