Math E-21b – Spring 2024 – Homework #7

Problem 1: Using paper and pencil, perform the Gram-Schmidt process on the given sequence of vectors and find the QR-factorization of the corresponding matrix.

(5.2/8) $\left\{ \begin{bmatrix} 5 \\ 4 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 7 \\ -2 \end{bmatrix} \right\}$ (5.2/22) $\begin{bmatrix} 5 & 3 \\ 4 & 6 \\ 2 & 7 \\ 2 & -2 \end{bmatrix}$

Problem 2: Using paper and pencil, perform the Gram-Schmidt process on the given sequence of vectors and find the QR-factorization of the corresponding matrix.

(5.2/14) $\left\{ \begin{bmatrix} 1 \\ 7 \\ 1 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 7 \\ 2 \\ 7 \end{bmatrix}, \begin{bmatrix} 1 \\ 8 \\ 1 \\ 6 \end{bmatrix} \right\}$ (5.2/28) $\begin{bmatrix} 1 & 0 & 1 \\ 7 & 7 & 8 \\ 1 & 2 & 1 \\ 7 & 7 & 6 \end{bmatrix}$

Problem 3: (5.2/34) Find an orthonormal basis of the kernel of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}$.

Problem 4: (5.3/32) (a) Consider an $n \times m$ matrix **A** such that $\mathbf{A}^{\mathrm{T}}\mathbf{A} = \mathbf{I}_m$. Is it necessarily true that $\mathbf{A}\mathbf{A}^{\mathrm{T}} = \mathbf{I}_n$? Explain.

(b) Consider an $n \times n$ matrix **A** such that $\mathbf{A}^{\mathrm{T}}\mathbf{A} = \mathbf{I}_n$. Is it necessarily true that $\mathbf{A}\mathbf{A}^{\mathrm{T}} = \mathbf{I}_n$? Explain.

Problem 5: (5.3/40) Consider the subspace *W* of $\mathbf{R}^4$ spanned by the vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 9 \\ -5 \\ 3 \end{bmatrix}$.

Find the matrix of the orthogonal projection onto *W*.

Problem 6: (5.3/44) Consider an $n \times m$ matrix **A**. Find $\dim(\operatorname{im}\mathbf{A}) + \dim(\ker(\mathbf{A}^{\mathrm{T}}))$ in terms of *m* and *n*.

Problem 7: (5.3/47) If $\mathbf{A} = \mathbf{QR}$ is a QR-factorization, what is the relationship between $\mathbf{A}^{\mathrm{T}}\mathbf{A}$ and $\mathbf{R}^{\mathrm{T}}\mathbf{R}$?

Problem 8: (5.3/48) Consider an invertible $n \times n$ matrix **A**. Can you write **A** as $\mathbf{A} = \mathbf{LQ}$, where **L** is a lower triangular matrix and **Q** is orthogonal? *Hint*: Consider the QR factorization of $\mathbf{A}^{\mathrm{T}}$.

Problem 9: (5.3/68) The formula $\mathbf{A}(\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}}$ for the matrix of an orthogonal projection is derived in Exercise 5.3/67 [and in section 5.4 and in the Lecture Notes]. Now consider the **QR** factorization of **A**, and express the matrix $\mathbf{A}(\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}}$ in terms of **Q**. [Note: The matrix **A** is constructed so that its columns form a basis for the subspace onto which vectors are being orthogonally projected.]

Problem 10: (5.4/6) If **A** is an $n \times m$ matrix, is the formula $\operatorname{im}(\mathbf{A}) = \operatorname{im}(\mathbf{A}\mathbf{A}^{\mathrm{T}})$ necessarily true? Explain.

Problem 11: (5.4/7) Consider a symmetric $n \times n$ matrix **A**. In this case, what is the relationship between $\operatorname{im}(\mathbf{A})$ and $\ker(\mathbf{A})$?

Problem 12: (5.4/10) Consider a consistent system $\mathbf{Ax} = \mathbf{b}$.

a. Show that this system has a solution $\mathbf{x}_0$ in $(\ker\mathbf{A})^{\perp}$. *Hint*: An arbitrary solution **x** of the system can be written as $\mathbf{x} = \mathbf{x}_h + \mathbf{x}_0$, where $\mathbf{x}_h$ is in $\ker\mathbf{A}$ and $\mathbf{x}_0$ is in $(\ker\mathbf{A})^{\perp}$.

b. Show that the system $\mathbf{Ax} = \mathbf{b}$ has only one solution in $(\ker\mathbf{A})^{\perp}$.
Hint: If $\mathbf{x}_0$ and $\mathbf{x}_1$ are two solutions in $(\ker\mathbf{A})^{\perp}$, think about $\mathbf{x}_1 - \mathbf{x}_0$.

c. If $\mathbf{x}_0$ is the solution in $(\ker\mathbf{A})^{\perp}$ and $\mathbf{x}_1$ is another solution of the system $\mathbf{Ax} = \mathbf{b}$, show that $\|\mathbf{x}_0\| < \|\mathbf{x}_1\|$.
The vector $\mathbf{x}_0$ is called the *minimal solution* of the linear system $\mathbf{Ax} = \mathbf{b}$.

Problem 13: (5.4/22) Find the least-squares solution $\mathbf{x}^*$ of the system $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 5 & 3 \\ 4 & 5 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 5 \\ 9 \\ 2 \end{bmatrix}$.

Determine the error $\left\| \mathbf{b} - \mathbf{Ax}^* \right\|$.

Problem 14: (5.4/26) Find the least-squares solution $\mathbf{x}^*$ of the system $\mathbf{Ax} = \mathbf{b}$, where

$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Problem 15: (5.4/32) Fit a quadratic polynomial to the data points (0, 27), (1, 0), (2, 0), (3, 0), using least squares. Sketch the solution.

Problem 16: (5.4/40) Consider the data in the following table: [we'll seek a relation of the form $D = ka^n$]

Planet	***a*** Mean Distance from the Sun (in Astronomical Units)	***D*** Period of Revolution (in Earth Years)
Mercury	0.387	0.241
Earth	1.000	1.000
Jupiter	5.203	11.86
Uranus	19.19	84.04
Pluto	39.53	248.6

(a) Using logarithms, fit a function of the form $\ln(D) = c + n\ln(a)$ to the data points $(\ln(a_i), \ln(D_i))$, using least squares.

(b) Use your answer in part (a) to fit a power function $D = ka^n$ to the data points (a_i, D_i).

(c) Explain in terms of Kepler's laws of planetary motion. Explain why the constant k is close to 1.

Problem 17: (5.4/41) In the accompanying table, we list the public debt *D* of the United States (in billions of dollars), in various years *t* (as of September 30).

year	1975	1985	1995	2005
D	533	1,823	4,974	7,933

a. Letting $t = 0$ in 1975, fit a linear function of the form $\log(D) = c_0 + c_1 t$ to the data points $(t_i, \log(D_i))$, using least squares. Use the result to fit an exponential function to the data points (t_i, D_i).

b. What debt does your formula in part (a) predict for the year 2015?

For additional practice (not to be turned in):

Section 5.2:

6. Using paper and pencil, perform the Gram-Schmidt process on the sequence of vectors $\left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix} \right\}$.

20. Using paper and pencil, find the QR-factorization of the matrix $\begin{bmatrix} 2 & 3 & 5 \\ 0 & 4 & 6 \\ 0 & 0 & 7 \end{bmatrix}$. [*Note*: This relates to #6.]

33. Find an orthonormal basis of the kernel of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$.

44. Consider an $n \times m$ matrix **A** with $\text{rank}(\mathbf{A}) < m$. Is it always possible to write **A** as $\mathbf{A} = \mathbf{QR}$ where **Q** is an $n \times m$ matrix with orthonormal columns and **R** is upper triangular? Explain.

45. Consider an $n \times m$ matrix **A** with $\text{rank}(\mathbf{A}) = m$. Is it always possible to write **A** as $\mathbf{A} = \mathbf{QL}$ where **Q** is an $n \times m$ matrix with orthonormal columns and **L** is a lower triangular $m \times m$ matrix with positive diagonal entries? Explain.

Section 5.3:
If the $n \times n$ matrices **A** and **B** are orthogonal matrices, which of the matrices in Exercises 5 through 11 must be orthogonal as well?

5. $3\mathbf{A}$ 6. $-\mathbf{B}$ 7. $\mathbf{AB}$ 8. $\mathbf{A}+\mathbf{B}$ 9. $\mathbf{B}^{-1}$ 10. $\mathbf{B}^{-1}\mathbf{AB}$ 11. $\mathbf{A}^{\mathrm{T}}$

31. Are the *rows* of an orthogonal matrix **A** necessarily orthonormal? [Explain why or why not.]

42. Let **A** be the matrix of an orthogonal projection. Find $\mathbf{A}^2$ in two ways:
   a. Geometrically. (Consider what happens when you apply an orthogonal projection twice.)
   b. By computation, using the formula given in Fact 5.3.10 (matrix of an orthogonal projection in terms of an orthonormal basis for a given subspace).

45. For which $n \times m$ matrices **A** does the equation $\dim(\ker(\mathbf{A})) = \dim(\ker(\mathbf{A}^{\mathrm{T}}))$ hold? Explain.

46. Consider a QR-factorization $\mathbf{M} = \mathbf{QR}$. Show that $\mathbf{R} = \mathbf{Q}^{\mathrm{T}}\mathbf{M}$.

Section 5.4:

15. Consider an $m \times n$ matrix **A** with $\ker(\mathbf{A}) = \{\mathbf{0}\}$. Show that there exists an $n \times m$ matrix **B** such that $\mathbf{BA} = \mathbf{I}_n$.
   Hint: $\mathbf{A}^{\mathrm{T}}\mathbf{A}$ is invertible.

17. Does the equation $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}^{\mathrm{T}}\mathbf{A})$ hold for all $n \times m$ matrices **A**? Explain.

18. Does the equation $\text{rank}(\mathbf{A}^{\mathrm{T}}\mathbf{A}) = \text{rank}(\mathbf{A}\mathbf{A}^{\mathrm{T}})$ hold for all $n \times m$ matrices **A**? Explain.
   Hint: Exercise 17 is useful.

24. Find the least-squares solution $\mathbf{x}^*$ of the system $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ 7 \end{bmatrix}$. Draw a sketch showing the vector **b**, the image of **A**, the vector $\mathbf{Ax}^*$, and the vector $\mathbf{b} - \mathbf{Ax}^*$.

37. The accompanying table lists several commercial airlines, the year they were introduced, and the number of displays in the cockpit.

Plane	**Year *t***	**Displays *d***
Douglas DC-3	1935	35
Lockheed Constellation	1946	46
Boeing 707	1959	77
Concorde	1969	133

   a. Fit a linear function of the form $\log(d) = c_0 + c_1 t$ to the data points $(t_i, \log(d_i))$, using least squares.
   b. Use your answer in part (a) to fit an exponential function $d = ka^t$ to the data points (t_i, d_i).
   c. The Airbus A320 was introduced in 1988. Based on your answer in part (b), how many displays do you expect in the cockpit of this plane? (There are 93 displays in the cockpit of an Airbus A320.) Explain.

38. In the accompanying table, we list the height *h*, the gender *g*, and the weight *w* of some young adults.

Height *h* (in inches above 5 feet)	**Gender *g*** (1 = "female", 0 = "male")	**Weight *w*** (in pounds)
2	1	110
12	0	180
5	1	120
11	1	160
6	0	160

Fit a function of the form $w = c_0 + c_1 h + c_2 g$ to these data, using least squares. Before you do the computations, think about the signs of c_1 and c_2. What signs would you expect if these data were representative of the general population? Why? What is the sign of c_0? What is the practical significance of c_0?

42. If **A** is any matrix, show that the linear transformation $L(\mathbf{x}) = \mathbf{Ax}$ from $\text{im}(\mathbf{A}^{\mathrm{T}})$ to $\text{im}(\mathbf{A})$ is an isomorphism. This provides yet another proof of the formula $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}^{\mathrm{T}})$.

Chapter 5 True/False Exercise

1. If A and B are symmetric $n \times n$ matrices, then $A + B$ must be symmetric as well.

2. If matrices A and S are orthogonal, then $S^{-1}AS$ is orthogonal as well.

3. All nonzero symmetric matrices are invertible.

4. If A is an $n \times n$ matrix such that $AA^T = I_n$, then A must be an orthogonal matrix.

5. If $\vec{u}$ is a unit vector in $\mathbb{R}^n$, and $L = \text{span}(\vec{u})$, then $\text{proj}_L(\vec{x}) = (\vec{x} \cdot \vec{u})\vec{x}$ for all vectors $\vec{x}$ in $\mathbb{R}^n$.

6. If A is a symmetric matrix, then $7A$ must be symmetric as well.

7. If T is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$ such that $T(\vec{e}_1), T(\vec{e}_2), \ldots, T(\vec{e}_n)$ are all unit vectors, then T must be an orthogonal transformation.

8. If A is an invertible matrix, then the equation $(A^T)^{-1} = (A^{-1})^T$ must hold.

9. If matrix A is orthogonal, then matrix A^2 must be orthogonal as well.

10. The equation $(AB)^T = A^T B^T$ holds for all $n \times n$ matrices A and B.

11. If matrix A is orthogonal, then A^T must be orthogonal as well.

12. If A and B are symmetric $n \times n$ matrices, then AB must be symmetric as well.

13. If matrices A and B commute, then A must commute with B^T as well.

14. If A is any matrix with $\ker(A) = \{\vec{0}\}$, then the matrix AA^T represents the orthogonal projection onto the image of A.

15. If A and B are symmetric $n \times n$ matrices, then $ABBA$ must be symmetric as well.

16. If matrices A and B commute, then matrices A^T and B^T must commute as well.

17. There exists a subspace V of $\mathbb{R}^5$ such that $\dim(V) = \dim(V^\perp)$, where $V^\perp$ denotes the orthogonal complement of V.

18. Every invertible matrix A can be expressed as the product of an orthogonal matrix and an upper triangular matrix.

19. If $\vec{x}$ and $\vec{y}$ are two vectors in $\mathbb{R}^n$, then the equation $\|\vec{x} + \vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2$ must hold.

20. The equation $\det(A^T) = \det(A)$ holds for all 2×2 matrices A.

21. If A and B are orthogonal 2×2 matrices, then $AB = BA$.

22. If A is a symmetric matrix, vector $\vec{v}$ is in the image of A, and $\vec{w}$ is in the kernel of A, then the equation $\vec{v} \cdot \vec{w} = 0$ must hold.

23. The formula $\ker(A) = \ker(A^T A)$ holds for all matrices A.

24. If $A^T A = AA^T$ for an $n \times n$ matrix A, then A must be orthogonal.

25. The determinant of all orthogonal 2×2 matrices is 1.

26. If A is any square matrix, then matrix $\frac{1}{2}(A - A^T)$ is skew-symmetric.

27. The entries of an orthogonal matrix are all less than or equal to 1.

28. Every nonzero subspace of $\mathbb{R}^n$ has an orthonormal basis.

29. $\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$ is an orthogonal matrix.

30. If V is a subspace of $\mathbb{R}^n$ and $\vec{x}$ is a vector in $\mathbb{R}^n$, then vector $\text{proj}_V \vec{x}$ must be orthogonal to vector $\vec{x} - \text{proj}_V \vec{x}$.

31. There exist orthogonal 2×2 matrices A and B such that $A + B$ is orthogonal as well.

32. If $\|A\vec{x}\| \le \|\vec{x}\|$ for all $\vec{x}$ in $\mathbb{R}^n$, then A must represent the orthogonal projection onto a subspace V of $\mathbb{R}^n$.

33. If A is an invertible matrix such that $A^{-1} = A$, then A must be orthogonal.

34. If the entries of two vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^n$ are all positive, then $\vec{v}$ and $\vec{w}$ must enclose an acute angle.

35. The formula $(\ker B)^\perp = \text{im}(B^T)$ holds for all matrices B.

36. The matrix $A^T A$ is symmetric for all matrices A.

37. If matrix A is similar to B and A is orthogonal, then B must be orthogonal as well.

38. The formula $\text{im}(B) = \text{im}(B^T B)$ holds for all square matrices B.

39. If matrix A is symmetric and matrix S is orthogonal, then matrix $S^{-1}AS$ must be symmetric.

40. If A is a square matrix such that $A^T A = AA^T$, then $\ker(A) = \ker(A^T)$.

41. Any square matrix can be written as the sum of a symmetric and a skew-symmetric matrix.

42. If $x_1, x_2, \ldots, x_n$ are any real numbers, then the inequality

$$\left(\sum_{k=1}^{n} x_k\right)^2 \leq n \sum_{k=1}^{n} (x_k^2)$$

must hold.

43. If $AA^T = A^2$ for a 2×2 matrix A, then A must be symmetric.

44. If V is a subspace of $\mathbb{R}^n$ and $\vec{x}$ is a vector in $\mathbb{R}^n$, then the inequality $\vec{x} \cdot (\text{proj}_V \vec{x}) \geq 0$ must hold.

45. If A is an $n \times n$ matrix such that $\|A\vec{u}\| = 1$ for all unit vectors $\vec{u}$, then A must be an orthogonal matrix.

46. If A is any symmetric 2×2 matrix, then there must exist a real number x such that matrix $A - xI_2$ fails to be invertible.

47. There exists a basis of $\mathbb{R}^{2\times 2}$ that consists of orthogonal matrices.

48. If $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, then the matrix Q in the QR factorization of A is a rotation matrix.

49. There exists a linear transformation L from $\mathbb{R}^{3\times 3}$ to $\mathbb{R}^{2\times 2}$ whose kernel is the space of all skew-symmetric 3×3 matrices.

50. If a 3×3 matrix A represents the orthogonal projection onto a plane V in $\mathbb{R}^3$, then there must exist an orthogonal 3×3 matrix S such that $S^T AS$ is diagonal.