Math E-21b – Spring 2025 – Homework #6

Section 5.1:

Problem 1. (5.1/12) Give an algebraic proof for the *triangle inequality* $\|\mathbf{v}+\mathbf{w}\| \le \|\mathbf{v}\|+\|\mathbf{w}\|$. Draw a sketch.

[Hint: Expand $\|\mathbf{v}+\mathbf{w}\|^2 = (\mathbf{v}+\mathbf{w})\cdot(\mathbf{v}+\mathbf{w})$. Then use the Cauchy-Schwarz inequality ($|\mathbf{v}\cdot\mathbf{w}| \le \|\mathbf{v}\|\|\mathbf{w}\|$).]

Problem 2. (5.1/15) Consider the vector $\mathbf{v}=\begin{bmatrix}1\\2\\3\\4\end{bmatrix}$ in $\mathbf{R}^4$. Find a basis of the subspace of $\mathbf{R}^4$ consisting of all vectors perpendicular (orthogonal) to $\mathbf{v}$.

Problem 3. (5.1/16) Consider the vectors $\mathbf{u}_1=\begin{bmatrix}1/2\\1/2\\1/2\\1/2\end{bmatrix}$, $\mathbf{u}_2=\begin{bmatrix}1/2\\1/2\\-1/2\\-1/2\end{bmatrix}$, $\mathbf{u}_3=\begin{bmatrix}1/2\\-1/2\\1/2\\-1/2\end{bmatrix}$ in $\mathbf{R}^4$. Can you find a vector $\mathbf{u}_4$ in $\mathbf{R}^4$ such that the vectors $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$, $\mathbf{u}_4$ are orthonormal? If so, how many such vectors are there?

Problem 4. (5.1/17) Find a basis for $W^{\perp}$, where $W = \text{span}\left\{\begin{bmatrix}1\\2\\3\\4\end{bmatrix},\begin{bmatrix}5\\6\\7\\8\end{bmatrix}\right\}$.

Problem 5. (5.1/18) Here is an infinite dimension version of Euclidean space: In the space of all infinite sequences, consider the subspace l_2 of square-summable sequences [i.e., those sequences $(x_1, x_2, \ldots)$ for which the infinite series $x_1^2 + x_2^2 + \cdots$ converges]. For $\mathbf{x}$ and $\mathbf{y}$ in l_2, we define $\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + \cdots}$ and $\mathbf{x}\cdot\mathbf{y} = x_1y_1 + x_2y_2 + \cdots$.

(Why does the series $x_1y_1 + x_2y_2 + \cdots$ converge?)

a. Check that $\mathbf{x} = (1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots)$ is in l_2, and find $\|\mathbf{x}\|$. Recall the formula for the geometric series:

$$1 + a + a^2 + a^3 + \cdots = \frac{1}{1-a}, \text{ if } -1 < a < 1.$$

b. Find the angle between $(1, 0, 0, \ldots)$ and $(1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots)$.

c. Give an example of a sequence $(x_1, x_2, \ldots)$ that converges to 0 (i.e., $\lim_{n\to\infty} x_n = 0$) but does not belong to l_2.

d. Let L be the subspace of l_2 spanned by $(1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots)$. Find the orthogonal projection of $(1, 0, 0, \ldots)$ onto L.

The Hilbert space l_2 was initially used mostly in physics: Werner Heisenberg's formulation of quantum mechanics is in terms of l_2. Today, this space is used in many other applications, including economics. (See, for example, the work of the economist Andreu Mas-Colell of the University of Barcelona.)

Note: Problems 6 and 7 (below) use the fact that if $\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k\}$ is an orthonormal basis for a subspace V of $\mathbf{R}^n$, then the orthogonal projection onto V of any vector $\mathbf{x}$ in $\mathbf{R}^n$ is given by
$\text{Proj}_V\mathbf{x} = (\mathbf{x}\cdot\mathbf{u}_1)\mathbf{u}_1 + \cdots + (\mathbf{x}\cdot\mathbf{u}_k)\mathbf{u}_k$.

Problem 6. (5.1/26)

Find the orthogonal projection of $\begin{bmatrix} 49 \\ 49 \\ 49 \end{bmatrix}$ onto the subspace of $\mathbf{R}^3$ spanned by $\begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ -6 \\ 2 \end{bmatrix}$.

Problem 7. (5.1/28)

Find the orthogonal projection of $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ onto the subspace of $\mathbf{R}^4$ spanned by $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$.

<u>Section 5.4</u>:

Problem 8. (5.4/2) Consider the subspace $\text{im}(\mathbf{A})$ of $\mathbf{R}^3$, where $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$. Find a basis of $\ker(\mathbf{A}^{\mathrm{T}})$, and draw a sketch illustrating the formula $(\text{im}\,\mathbf{A})^{\perp} = \ker(\mathbf{A}^{\mathrm{T}})$ in this case.

Problem 9. (5.4/4) Let **A** be an $n \times m$ matrix. Is the formula $(\ker \mathbf{A})^{\perp} = \text{im}(\mathbf{A}^{\mathrm{T}})$ necessarily true? Explain.

Problem 10. (5.4/5)

Let V be the solution space of the linear system $\left\{ \begin{matrix} x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 + 2x_2 + 5x_3 + 4x_4 = 0 \end{matrix} \right\}$. Find a basis for $\mathrm{V}^{\perp}$.

Problem 11. (5.4/16) Use the formula $(\text{im}\,\mathbf{A})^{\perp} = \ker(\mathbf{A}^{\mathrm{T}})$ to prove the equation $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}^{\mathrm{T}})$.

Problem 12. For the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 1 & 0 \\ 1 & 3 & 4 \end{bmatrix}$, find a basis for <u>each</u> of the ***Four Fundamental Subspaces***:

(a) $\text{im}(\mathbf{A})$ (b) $\ker(\mathbf{A})$ (c) $\text{im}(\mathbf{A}^{\mathrm{T}})$ (d) $\ker(\mathbf{A}^{\mathrm{T}})$

For additional practice:

Section 5.1:

3. Find the length of the vector $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$.

5. Find the angle between the vectors $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$.

23. Prove Fact 5.1.8d: $(\mathrm{V}^{\perp})^{\perp} = \mathrm{V}$ for any subspace V of $\mathbf{R}^n$. Hint: Show that $\mathrm{V} \subseteq (\mathrm{V}^{\perp})^{\perp}$ by the definition of $\mathrm{V}^{\perp}$; then show that $\dim(\mathrm{V}) = \dim[(\mathrm{V}^{\perp})^{\perp}]$, by Fact 5.1.8c, i.e. that the dimensions of V and $\mathrm{V}^{\perp}$ sum to *n*.

29. Consider the orthonormal vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5$ in $\mathbf{R}^{10}$.
Find the length of the vector $\mathbf{x} = 7\mathbf{u}_1 - 3\mathbf{u}_2 + 2\mathbf{u}_3 + \mathbf{u}_4 - \mathbf{u}_5$.

Section 5.4:

1. Consider the subspace $\text{im}(\mathbf{A})$ of $\mathbf{R}^2$, where $\mathbf{A} = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$. Find a basis of $\ker(\mathbf{A}^{\mathrm{T}})$, and draw a sketch illustrating the formula $(\text{im}\,\mathbf{A})^{\perp} = \ker(\mathbf{A}^{\mathrm{T}})$ in this case.