Math E-21b – Spring 2025 – Homework #5

Problem 1. (3.4/60) Is the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ similar to the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$?

[**Definition**: Two matrices **A** and **B** are called ***similar*** if $\mathbf{B} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$ for some invertible matrix **S**, i.e. they represent the same linear transformation relative to different bases.]

Problem 2. (3.4/62) Find a basis $\mathscr{B}$ of $\mathbf{R}^2$ such that the $\mathscr{B}$-matrix of the linear transformation $T(\mathbf{x}) = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\mathbf{x}$ is $\mathbf{B} = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$.

Problem 3. (3.4/70) Is there a basis $\mathscr{B}$ of $\mathbf{R}^2$ such that the $\mathscr{B}$-matrix **B** of the linear transformation $T(\mathbf{x}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\mathbf{x}$ is upper triangular? *Hint*: Think about the first column of **B**.

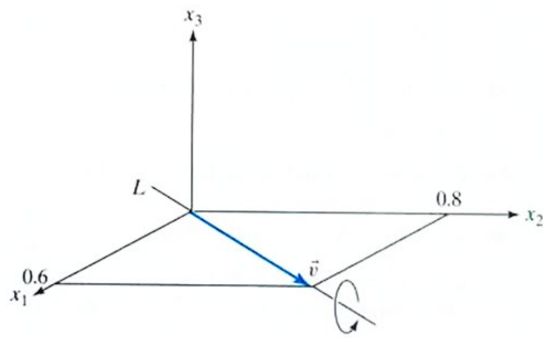
Problem 4. (3.4/71) Suppose the matrix **A** is similar to **B**, with $\mathbf{B} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$.

a. Show that if **x** is in ker(**B**), then **Sx** is in ker(**A**).

b. Show that nullity(**A**) = nullity(**B**). *Hint*: If $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p\}$ is a basis for ker(**B**), then the vectors $\{\mathbf{S}\mathbf{v}_1, \mathbf{S}\mathbf{v}_2, \cdots, \mathbf{S}\mathbf{v}_p\}$ in ker(**A**) are linearly independent. Now reverse the roles of **A** and **B**.

c. (3.4/72) If **A** is similar to **B**, what is the relationship between rank(**A**) and rank(**B**)? See exercise 71.

Problem 5. (3.4/73) Let L be the line in $\mathbf{R}^3$ spanned by the vector $\mathbf{v} = \begin{bmatrix} 0.6 \\ 0.8 \\ 0 \end{bmatrix}$. Let T from $\mathbf{R}^3$ to $\mathbf{R}^3$ be the rotation about this line through an angle of $\pi/2$, in the direction indicated in the accompanying sketch.
Find the matrix **A** such that $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$.

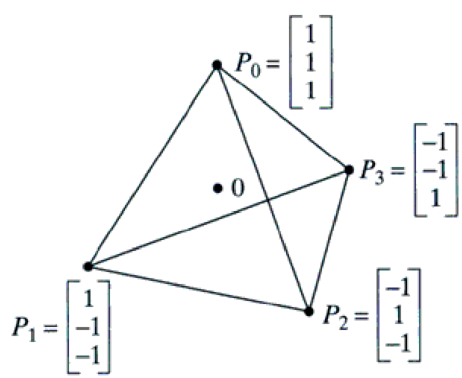


Problem 6. (3.4/74) Consider the regular tetrahedron in the accompanying sketch whose center is at the origin. Let $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be the position vectors of the four vertices of the tetrahedron:
$\mathbf{v}_0 = \overrightarrow{OP_0}$, $\mathbf{v}_1 = \overrightarrow{OP_1}$, $\mathbf{v}_2 = \overrightarrow{OP_2}$, $\mathbf{v}_3 = \overrightarrow{OP_3}$.

a. Find the sum $\mathbf{v}_0 + \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$.

b. Find the coordinate vector of $\mathbf{v}_0$ with respect to the basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

c. Let T be the linear transformation with $T(\mathbf{v}_0) = \mathbf{v}_3$, $T(\mathbf{v}_3) = \mathbf{v}_1$, and $T(\mathbf{v}_1) = \mathbf{v}_0$. What is $T(\mathbf{v}_2)$? Describe the transformation T geometrically (as a reflection, rotation, projection, or whatever). Find the matrix **B** of T with respect to the basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. What is $\mathbf{B}^3$? Explain.



Note: The same figure was used in HW3, Problem 6.

Find a basis for each of the spaces in Problems 7-9 and determine its dimension.

Problem 7. (4.1/20) The space of all matrices $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in $\mathbf{R}^{2\times 2}$ such that $a + d = 0$.

Problem 8. (4.1/26) The space of all polynomials $f(t)$ in P_3 such that $f(1) = 0$ and $\int_{-1}^{1} f(t)\,dt = 0$.

Problem 9. (4.1/30) The space of all 2×2 matrices **A** such that $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Problem 10. a) (4.2/6) Is the transformation $T(\mathbf{M}) = \mathbf{M}\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ from $\mathbf{R}^{2\times 2}$ to $\mathbf{R}^{2\times 2}$ linear?

If it is, determine whether it is an isomorphism.

b) (4.2/52) Find the rank and nullity, and bases for the image and kernel of this transformation.

Problem 11. a) (4.2/25) Is the transformation $[T(f)](t) = f''(t) + 4f'(t)$ from P_2 to P_2 linear? If it is, determine whether it is an isomorphism.

b) (4.2/53) Find the image, rank, kernel and nullity of this transformation.

c) (4.3/22) Find the matrix of the linear transformation $T(f) = f'' + 4f'$ from P_2 to P_2 relative to the basis $\mathcal{U} = \{1, t, t^2\}$.

Problem 12. (4.2/66) Find the kernel and nullity of the transformation $T(f) = f - f'$ from C^∞ to C^∞.
[C^∞ denotes the linear space consisting of all infinitely differentiable functions of one variable.]

Problem 13. a) (4.3/27) Find the matrix **A** of the linear transformation $[T(f)](t) = f(2t-1)$ from P_2 to P_2 relative to the basis $\mathcal{U} = \{1, t, t^2\}$.

b) (4.3/28) Find the matrix **B** of the linear transformation $[T(f)](t) = f(2t-1)$ from P_2 to P_2 relative to the basis $\mathcal{B} = \{1, t-1, (t-1)^2\}$.

c) (4.3/47) Find the change of basis matrix **S** from the basis $\mathcal{B} = \{1, t-1, (t-1)^2\}$ to the "standard" basis $\mathcal{U} = \{1, t, t^2\}$ of P_2. That is, find **S** such that $[f]_{\mathcal{U}} = \mathbf{S}[f]_{\mathcal{B}}$ for any element $f \in P_2$.

d) Verify the formula $\mathbf{SB} = \mathbf{AS}$ (that is, $\mathbf{B} = \mathbf{S}^{-1}\mathbf{AS}$) for the matrices **B** and **A** you found in parts a) and b).

e) Find the change of basis matrix **Q** from $\mathcal{U}$ to $\mathcal{B}$, i.e. such that $[f]_{\mathcal{B}} = \mathbf{Q}[f]_{\mathcal{U}}$ for any element $f \in P_2$.

For additional practice:

Section 3.4:

59. Is the matrix $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ similar to the matrix $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$?

69. If **A** is a 2×2 matrix such that $\mathbf{A}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$ and $\mathbf{A}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$, show that **A** is similar to a diagonal matrix **D**. Find an invertible matrix **S** such that $\mathbf{S}^{-1}\mathbf{AS} = \mathbf{D}$.

Section 4.1:

Which of the subsets of P_2 given in Exercises 1, 2, and 3 are subspaces of P_2? Find a basis for those that are subspaces. [P_2 is the linear space consisting of polynomials of degree less than or equal to 2.]

1. $\{p(t) : p(0) = 2\}$ 2. $\{p(t) : p(0) = 0\}$ 3. $\{p(t) : p'(1) = p(2)\}$ (p' denotes the derivative.)

Which of the subsets of $\mathbf{R}^{3\times3}$ such given in Exercises 9, 10, and 11 are subspaces of $\mathbf{R}^{3\times3}$?

9. The 3 × 3 matrices whose entries are all greater than or equal to zero.

10. The 3×3 matrices **A** such that the vector $\begin{bmatrix}1\\2\\3\end{bmatrix}$ is in the kernel of **A**.

11. The 3×3 matrices in reduced row-echelon form.

Find a basis for each of the spaces in Exercises 25 and 29 and determine its dimension.

25. The space of all polynomials $f(t)$ in P_2 such that $f(1)=0$.

29. The space of all 2×2 matrices **A** such that $\mathbf{A}\begin{bmatrix}1&1\\1&1\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$.

Section 4.2:

2. Is the transformation $T(\mathbf{M}) = 7\mathbf{M}$ from $\mathbf{R}^{2\times2}$ to $\mathbf{R}^{2\times2}$ linear? If so, determine whether it is an isomorphism.

4. Is the transformation $T(\mathbf{M}) = \det(\mathbf{M})$ from $\mathbf{R}^{2\times2}$ to **R** linear? If so, determine whether it is an isomorphism.

67. For which constants *k* is the linear transformation $T(\mathbf{M})=\begin{bmatrix}2&3\\0&4\end{bmatrix}\mathbf{M}-\mathbf{M}\begin{bmatrix}3&0\\0&k\end{bmatrix}$ an isomorphism from $\mathbf{R}^{2\times2}$ to $\mathbf{R}^{2\times2}$.

81. In this exercise, we will outline a proof of the Rank-Nullity Theorem: If *T* is a linear transformation from *V* to *W*, where *V* is finite-dimensional, then dim(*V*) = dim(im *T*) + dim(ker *T*) = rank(*T*) + nullity(*T*).

a. Explain why ker(*T*) and image(*T*) are finite dimensional. *Hint*: Use Exercises 4.1.54 and 4.1.57.

Now, consider a basis $\{\mathbf{v}_1,\mathbf{v}_2,\cdots,\mathbf{v}_n\}$ of ker(*T*), where *n* = nullity(*T*), and a basis $\{\mathbf{w}_1, \mathbf{v}_2,\ldots, \mathbf{w}_r\}$ of im(*T*), where *r* = rank(*T*). Consider vectors $\{\mathbf{u}_1, \mathbf{u}_2,\ldots, \mathbf{u}_r\}$ in *V* such that $T(\mathbf{u}_i)=\mathbf{w}_i$ for *i* = 1,…,*r*. Our goal is to show that the *r* + *n* vectors $\mathbf{u}_1, \mathbf{u}_2,\ldots, \mathbf{u}_r, \mathbf{v}_1, \mathbf{v}_2,\ldots, \mathbf{v}_n$ form a basis of *V*. This will prove our claim.

b. Show that the vectors $\mathbf{u}_1, \mathbf{u}_2,\ldots, \mathbf{u}_r, \mathbf{v}_1, \mathbf{v}_2,\ldots, \mathbf{v}_n$ are linearly independent. *Hint*: Consider a relation $c_1\mathbf{u}_1+\cdots c_r\mathbf{u}_r+d_1\mathbf{v}_1+\cdots d_n\mathbf{v}_n=\mathbf{0}$, apply linear transformation *T* to both sides, and take it from there.

c. Show that the vectors $\mathbf{u}_1, \mathbf{u}_2,\ldots, \mathbf{u}_r, \mathbf{v}_1, \mathbf{v}_2,\ldots, \mathbf{v}_n$ span *V*. *Hint*: Consider an arbitrary vector **v** in *V*, and write $T(\mathbf{v})=d_1\mathbf{w}_1+\cdots d_r\mathbf{w}_r$. Now show that the vector $\mathbf{v}-d_1\mathbf{u}_1+\cdots d_r\mathbf{u}_r$ is in the kernel of *T*, so that $\mathbf{v}-d_1\mathbf{u}_1+\cdots d_r\mathbf{u}_r$ can be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2,\ldots, \mathbf{v}_n$.

Section 4.3:

1. Are the polynomials $f(t)=7+3t+t^2$, $g(t)=9+9t+4t^2$, and $h(t)=3+2t+t^2$ linearly independent?

13. Find the matrix of the linear transformation $T(\mathbf{M})=\begin{bmatrix}1&1\\2&2\end{bmatrix}\mathbf{M}$ from $\mathbf{R}^{2\times2}$ to $\mathbf{R}^{2\times2}$ with respect to the basis

$\mathcal{U}=\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&1\\0&0\end{bmatrix},\begin{bmatrix}0&0\\1&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}$.

14. Find the matrix of the linear transformation $T(\mathbf{M})=\begin{bmatrix}1&1\\2&2\end{bmatrix}\mathbf{M}$ from $\mathbf{R}^{2\times2}$ to $\mathbf{R}^{2\times2}$ with respect to the basis

$\mathcal{B}=\left\{\begin{bmatrix}1&0\\-1&0\end{bmatrix},\begin{bmatrix}0&1\\0&-1\end{bmatrix},\begin{bmatrix}1&0\\2&0\end{bmatrix},\begin{bmatrix}0&1\\0&2\end{bmatrix}\right\}$.

44. a. Find the change of basis matrix **S** from the basis $\mathcal{B}$ considered in Exercise 14 to the standard basis $\mathcal{U}$ of $\mathbf{R}^{2\times2}$ considered in Exercise 13.

b. Verify the formula **SB** = **AS** (that is, $\mathbf{B} = \mathbf{S}^{-1}\mathbf{AS}$) for the matrices **B** and **A** you found in Exercises 14 and 13, respectively.

TRUE or FALSE?

1. The space $\mathbb{R}^{2\times 3}$ is 5-dimensional.
2. If $f_1, \ldots, f_n$ is a basis of a linear space V, then any element of V can be written as a linear combination of $f_1, \ldots, f_n$.
3. The space P_1 is isomorphic to $\mathbb{C}$.
4. If the kernel of a linear transformation T from P_4 to P_4 is $\{0\}$, then T must be an isomorphism.
5. If W_1 and W_2 are subspaces of a linear space V, then the intersection $W_1 \cap W_2$ must be a subspace of V as well.
6. If T is a linear transformation from P_6 to $\mathbb{R}^{2\times 2}$, then the kernel of T must be 3-dimensional.
7. The polynomials of degree less than 7 form a 7-dimensional subspace of the linear space of all polynomials.
8. The function $T(f) = 3f - 4f'$ from C^∞ to C^∞ is a linear transformation.
9. The lower triangular 2×2 matrices form a subspace of the space of all 2×2 matrices.
10. The kernel of a linear transformation is a subspace of the domain.
11. The linear transformation $T(f) = f + f''$ from C^∞ to C^∞ is an isomorphism.
12. All linear transformations from P_3 to $\mathbb{R}^{2\times 2}$ are isomorphisms.
13. If T is a linear transformation from V to V, then the intersection of $\operatorname{im}(T)$ and $\ker(T)$ must be $\{0\}$.
14. The space of all upper triangular 4×4 matrices is isomorphic to the space of all lower triangular 4×4 matrices.
15. Every polynomial of degree 3 can be expressed as a linear combination of the polynomial $(t-3)$, $(t-3)^2$, and $(t-3)^3$.
16. If a linear space V can be spanned by 10 elements, then the dimension of V must be ≤ 10.
17. The function $T(M) = \det(M)$ from $\mathbb{R}^{2\times 2}$ to $\mathbb{R}$ is a linear transformation.
18. There exists a 2×2 matrix A such that the space of all matrices commuting with A is 1-dimensional.
19. All bases of P_3 contain at least one polynomial of degree ≤ 2.
20. If T is an isomorphism, then T^{-1} must be an isomorphism as well.
21. If the image of a linear transformation T from P to P is all of P, then T must be an isomorphism.
22. If f_1, f_2, f_3 is a basis of a linear space V, then f_1, $f_1 + f_2$, $f_1 + f_2 + f_3$ must be a basis of V as well.
23. If a, b, and c are distinct real numbers, then the polynomials $(x-b)(x-c)$, $(x-a)(x-c)$, and $(x-a)(x-b)$ must be linearly independent.
24. The linear transformation $T(f(t)) = f(4t-3)$ from P to P is an isomorphism.
25. The linear transformation $T(M) = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} M$ from $\mathbb{R}^{2\times 2}$ to $\mathbb{R}^{2\times 2}$ has rank 1.
26. If the matrix of a linear transformation T (with respect to some basis) is $\begin{bmatrix} 3 & 5 \\ 0 & 4 \end{bmatrix}$, then there must exist a nonzero element f in the domain of T such that $T(f) = 3f$.
27. The kernel of the linear transformation $T(f(t)) = f(t^2)$ from P to P is $\{0\}$.
28. If S is any invertible 2×2 matrix, then the linear transformation $T(M) = SMS$ is an isomorphism from $\mathbb{R}^{2\times 2}$ to $\mathbb{R}^{2\times 2}$.
29. There exists a 2×2 matrix A such that the space of all matrices commuting with A is 2-dimensional.
30. There exists a basis of $\mathbb{R}^{2\times 2}$ that consists of four invertible matrices.
31. If W is a subspace of V, and if W is finite dimensional, then V must be finite dimensional as well.
32. There exists a linear transformation from $\mathbb{R}^{3\times 3}$ to $\mathbb{R}^{2\times 2}$ whose kernel consists of all lower triangular 3×3 matrices, while the image consists of all upper triangular 2×2 matrices.
33. Every two-dimensional subspace of $\mathbb{R}^{2\times 2}$ contains at least one invertible matrix.
34. If $\mathfrak{A} = (f, g)$ and $\mathfrak{B} = (f, f+g)$ are two bases of a linear space V, then the change of basis matrix from $\mathfrak{A}$ to $\mathfrak{B}$ is $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.
35. If the matrix of a linear transformation T with respect to a basis (f, g) is $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, then the matrix of T with respect to the basis (g, f) is $\begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$.
36. The linear transformation $T(f) = f'$ from P_n to P_n has rank n, for all positive integers n.
37. If the matrix of a linear transformation T (with respect to some basis) is $\begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}$, then T must be an isomorphism.
38. There exists a subspace of $\mathbb{R}^{3\times 4}$ that is isomorphic to P_9.
39. There exist two distinct subspaces W_1 and W_2 of $\mathbb{R}^{2\times 2}$ whose union $W_1 \cup W_2$ is a subspace of $\mathbb{R}^{2\times 2}$ as well.
40. There exists a linear transformation from P to P_5 whose image is all of P_5.
41. If $f_1, \ldots, f_n$ are polynomials such that the degree of f_k is k (for $k = 1, \ldots, n$), then $f_1, \ldots, f_n$ must be linearly independent.
42. The transformation $D(f) = f'$ from C^∞ to C^∞ is an isomorphism.
43. If T is a linear transformation from P_4 to W with $\operatorname{im}(T) = W$, then the inequality $\dim(W) \leq 5$ must hold.

44. The kernel of the linear transformation

$$T(f(t)) = \int_0^1 f(t)\,dt$$

from P to $\mathbb{R}$ is finite dimensional.

45. If T is a linear transformation from V to V, then $\{f \text{ in } V : T(f) = f\}$ must be a subspace of V.

46. If T is a linear transformation from P_6 to P_6 that transforms t^k into a polynomial of degree k (for $k = 1, \ldots, 6$), then T must be an isomorphism.

47. There exist invertible 2×2 matrices P and Q such that the linear transformation $T(M) = PM - MQ$ is an isomorphism.

48. There exists a linear transformation from P_6 to $\mathbb{C}$ whose kernel is isomorphic to $\mathbb{R}^{2\times 2}$.

49. If f_1, f_2, f_3 is a basis of a linear space V, and if f is any element of V, then the elements $f_1 + f$, $f_2 + f$, $f_3 + f$ must form a basis of V as well.

50. There exists a two-dimensional subspace of $\mathbb{R}^{2\times 2}$ whose nonzero elements are all invertible.

51. The space P_{11} is isomorphic to $\mathbb{R}^{3\times 4}$.

52. If T is a linear transformation from V to W, and if both $\text{im}(T)$ and $\ker(T)$ are finite dimensional, then W must be finite dimensional.

53. If T is a linear transformation from V to $\mathbb{R}^{2\times 2}$ with $\ker(T) = \{0\}$, then the inequality $\dim(V) \leq 4$ must hold.

54. The function

$$T(f(t)) = \frac{d}{dt}\int_2^{3t+4} f(x)\,dx$$

from P_5 to P_5 is an isomorphism.

55. Any 4-dimensional linear space has infinitely many 3-dimensional subspaces.

56. If the matrix of a linear transformation T (with respect to some basis) is $\begin{bmatrix} 3 & 5 \\ 0 & 4 \end{bmatrix}$, then there must exist a nonzero element f in the domain of T such that $T(f) = 4f$.

57. If the image of a linear transformation T is infinite dimensional, then the domain of T must be infinite dimensional.

58. There exists a 2×2 matrix A such that the space of all matrices commuting with A is 3-dimensional.

59. If A, B, C, and D are noninvertible 2×2 matrices, then the matrices AB, AC, and AD must be linearly dependent.

60. There exist two distinct 3-dimensional subspaces W_1 and W_2 of P_4 such that the union $W_1 \cup W_2$ is a subspace of P_4 as well.

61. If the elements $f_1, \ldots, f_n$ (where $f_1 \neq 0$) are linearly dependent, then one element f_k can be expressed *uniquely* as a linear combination of the preceding elements $f_1, \ldots, f_{k-1}$.

62. There exists a 3×3 matrix P such that the linear transformation $T(M) = MP - PM$ from $\mathbb{R}^{3\times 3}$ to $\mathbb{R}^{3\times 3}$ is an isomorphism.

63. If f_1, f_2, f_3, f_4, f_5 are elements of a linear space V, and if there are exactly two redundant elements in the list f_1, f_2, f_3, f_4, f_5, then there must be exactly two redundant elements in the list f_2, f_4, f_5, f_1, f_3 as well.

64. There exists a linear transformation T from P_6 to P_6 such that the kernel of T is isomorphic to the image of T.

65. If T is a linear transformation from V to W, and if both $\text{im}(T)$ and $\ker(T)$ are finite dimensional, then V must be finite dimensional.

66. If the matrix of a linear transformation T (with respect to some basis) is $\begin{bmatrix} 3 & 5 \\ 0 & 4 \end{bmatrix}$, then there must exist a nonzero element f in the domain of T such that $T(f) = 5f$.

67. Every three-dimensional subspace of $\mathbb{R}^{2\times 2}$ contains at least one invertible matrix.