Math E-21b – Spring 2024 – Homework #4

Section 3.2:

Problem 1. (3.2/18) Use paper and pencil to identify the redundant vectors in $\begin{bmatrix}1\\1\\1\\1\end{bmatrix},\begin{bmatrix}1\\2\\3\\4\end{bmatrix},\begin{bmatrix}1\\4\\7\\10\end{bmatrix}$.

Thus determine whether the given vectors are linearly independent.

Problem 2. (3.2/28). Find a basis for the image of the matrix $\mathbf{A}=\begin{bmatrix}1&1&1\\1&2&5\\1&3&7\end{bmatrix}$.

Problem 3. (3.2/36) Consider a linear transformation T from $\mathbf{R}^n$ to $\mathbf{R}^p$ and some linearly dependent vectors $\mathbf{v}_1,\mathbf{v}_2,\dots,\mathbf{v}_m$ in $\mathbf{R}^n$. Are the vectors $T(\mathbf{v}_1),T(\mathbf{v}_2),\dots,T(\mathbf{v}_m)$ necessarily linearly dependent? How can you tell?

Problem 4. (3.2/40) Consider an $n\times p$ matrix **A** and a $p\times m$ matrix **B**. We are told that the columns of **A** and the columns of **B** are linearly independent (respectively). Are the columns of the product **AB** linearly independent as well? *Hint*: Exercise 3.1.51 is useful. [This exercise reads: "Consider an $n\times p$ matrix **A** and a $p\times m$ matrix **B** such that $\ker(\mathbf{A})=\{\mathbf{0}\}$ and $\ker(\mathbf{B})=\{\mathbf{0}\}$. Find $\ker(\mathbf{AB})$."]

Problem 5. (3.2/48) Express the plane V in $\mathbf{R}^3$ with equation $3x_1+4x_2+5x_3=0$ as the kernel of a matrix **A** and as the image of a matrix **B**. [***Note***: This exercise doesn't specify the sizes of the matrices **A** and **B**. There are many possible solutions, including the case where both **A** and **B** are 3×3 matrices. Think geometrically!]

Section 3.3:

Problem 6. (3.3/24) Find the reduced row-echelon form of the matrix $\mathbf{A}=\begin{bmatrix}4&8&1&1&6\\3&6&1&2&5\\2&4&1&9&10\\1&2&3&2&0\end{bmatrix}$.

Then find a basis of the image of **A** and a basis of the kernel of **A**.

Problem 7. (3.3/30) Find a basis of the subspace of $\mathbf{R}^4$ defined by the equation $2x_1-x_2+2x_3+4x_4=0$.

Problem 8. (3.3/32) Find a basis of the subspace of $\mathbf{R}^4$ that consists of all vectors perpendicular to both $\begin{bmatrix}1\\0\\-1\\1\end{bmatrix}$ and $\begin{bmatrix}0\\1\\2\\3\end{bmatrix}$.

Problem 9. (3.3/36) Can you find a 3×3 matrix **A** such that $\text{im}(\mathbf{A})=\ker(\mathbf{A})$? Explain.

Section 3.4:

In Problems 10 and 11, determine whether the vector **x** is in the span V of the vectors $\mathbf{v}_1,\mathbf{v}_2,\dots,\mathbf{v}_m$ (proceed "by inspection" if possible, and use the reduced row-echelon form if necessary). If **x** is in V, find the coordinates of **x** with respect to the basis $\mathcal{B}=\{\mathbf{v}_1,\mathbf{v}_2,\dots,\mathbf{v}_m\}$ of V, and write the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$.

Problem 10. (3.4/6) $\mathbf{x}=\begin{bmatrix}-4\\4\end{bmatrix}$; $\mathbf{v}_1=\begin{bmatrix}1\\2\end{bmatrix},\mathbf{v}_2=\begin{bmatrix}5\\6\end{bmatrix}$

Problem 11. (3.4/8) $\mathbf{x}=\begin{bmatrix}2\\3\\4\end{bmatrix}$; $\mathbf{v}_1=\begin{bmatrix}1\\1\\0\end{bmatrix},\mathbf{v}_2=\begin{bmatrix}2\\0\\1\end{bmatrix}$

In Problems 12 and 13, find the matrix **B** of the linear transformation $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$ with respect to the basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$.

Problem 12. (3.4/26) $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$; $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Problem 13. (3.4/28) $\mathbf{A} = \begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix}$; $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$; $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$; $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$

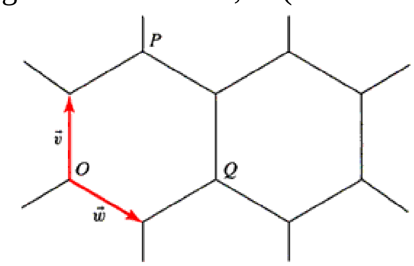
Problem 14. (3.4/42) Find a basis $\mathcal{B}$ of $\mathbf{R}^3$ such that the $\mathcal{B}$-matrix **B** of the linear transformation given by reflection *T* about the plane $x_1 - 2x_2 + 2x_3 = 0$ in $\mathbf{R}^3$ is diagonal. [Think geometrically!]

Problem 15. (3.4/44) Consider the plane $2x_1 - 3x_2 + 4x_3 = 0$ with basis $\mathcal{B}$ consisting of vectors $\begin{bmatrix} 8 \\ 4 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$. If $\mathbf{x}_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, find **x**.

Problem 16. (3.4/46) Consider the plane with equation $x_1 + 2x_2 + x_3 = 0$.

Find a basis $\mathcal{B}$ of this plane such that $\mathbf{x}_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ for $\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

Problem 17. (3.4/50) Given a <u>regular</u> hexagonal tiling of the plane, such as you might find on a kitchen floor, consider the basis $\mathcal{B}$ of $\mathbf{R}^2$ consisting of the vectors $\vec{v}$, $\vec{w}$ (of the same length) in the following sketch:



a. Find the coordinate vectors $\left[\overrightarrow{OP}\right]_{\mathcal{B}}$ and $\left[\overrightarrow{OQ}\right]_{\mathcal{B}}$.

Hint: Sketch the coordinate grid defined by the basis $\mathcal{B} = \{\vec{v}, \vec{w}\}$.

b. We are told that $\left[\overrightarrow{OR}\right]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$. Sketch the point *R*. Is *R* the vertex or the center of a tile?

c. We are told that $\left[\overrightarrow{OS}\right]_{\mathcal{B}} = \begin{bmatrix} 17 \\ 13 \end{bmatrix}$. Is *S* the center or the vertex of a tile?

Problem 18. (3.4/56) Find a basis $\mathcal{B}$ of $\mathbf{R}^2$ such that for the vectors $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ we have

$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ and $[\mathbf{y}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

Problem 19. (3.4/58) Consider a 3×3 matrix **A** and a vector **v** in $\mathbf{R}^3$ such that $\mathbf{A}^3\mathbf{v} = \mathbf{0}$, but $\mathbf{A}^2\mathbf{v} \neq \mathbf{0}$.

a. Show that the vectors $\{\mathbf{A}^2\mathbf{v}, \mathbf{A}\mathbf{v}, \mathbf{v}\}$ form a basis of $\mathbf{R}^3$. *Hint*: It suffices to show linear independence. Consider a relation $c_1\mathbf{A}^2\mathbf{v} + c_2\mathbf{A}\mathbf{v} + c_3\mathbf{v} = \mathbf{0}$ and multiply by $\mathbf{A}^2$ to show that $c_3 = 0$.

b. Find the matrix of the transformation $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$ with respect the basis $\{\mathbf{A}^2\mathbf{v}, \mathbf{A}\mathbf{v}, \mathbf{v}\}$.

For additional practice (not to be turned in):

Section 3.2:

Which of the sets W in Exercises 1 through 3 are subspaces of $\mathbf{R}^3$?

1. $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x+y+z=1 \right\}$ 2. $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x \le y \le z \right\}$ 3. $W = \left\{ \begin{bmatrix} x+2y+3z \\ 4x+5y+6z \\ 7x+8y+9z \end{bmatrix} : x, y, z \text{ arbitrary constants} \right\}$

6. Consider two subspaces V and W of $\mathbf{R}^n$.
   a. Is the intersection $V \cap W$ necessarily a subspace of $\mathbf{R}^n$?
   b. Is the union $V \cup W$ necessarily a subspace of $\mathbf{R}^n$?

In Exercises 17 and 19, use paper and pencil to identify the redundant vectors. Thus determine whether the given vectors are linearly independent.

17. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$ 19. $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 5 \\ 0 \end{bmatrix}$

24. Find a redundant column vector of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 3 & 6 \\ 1 & 2 & 5 \\ 1 & 1 & 4 \end{bmatrix}$, and write it as a linear combination of preceding columns. Use this representation to write a nontrivial relation among the columns, and thus find a nonzero vector in the kernel of **A**.

37. Consider a linear transformation T from $\mathbf{R}^n$ to $\mathbf{R}^p$ and some linearly independent vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ in $\mathbf{R}^n$. Are the vectors $T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_m)$ necessarily linearly independent? How can you tell?

41. Consider an $m \times n$ matrix **A** and a $n \times m$ matrix **B** (with $n \ne m$) such that $\mathbf{AB} = \mathbf{I}_m$. (We say that **A** is a *left inverse* of **B**.) Are the columns of **B** linearly independent? What about the columns of **A**?

49. Express the line L in $\mathbf{R}^3$ spanned by the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ as the image of a matrix **A** and as the kernel of a matrix **B**. [*Note*: This exercise doesn't specify the sizes of the matrices **A** and **B**. There are many possible solutions, including the case where both **A** and **B** are 3×3 matrices. Think geometrically!]

Section 3.3:

23. Find the reduced row-echelon form of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 3 & 4 & -6 & 8 \\ 0 & -1 & 3 & 1 \end{bmatrix}$.

    Then find a basis of the image of **A** and a basis of the kernel of **A**.

27. Determine whether the vectors $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 4 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 4 \\ -8 \end{bmatrix} \right\}$ form a basis of $\mathbf{R}^4$.

29. Find a basis of the subspace of $\mathbf{R}^3$ defined by the equation $2x_1 + 3x_2 + x_3 = 0$.

60. Consider two subspaces V and W of $\mathbf{R}^n$, where V is contained in W. Explain why $\dim(V) \le \dim(W)$. (This statement seems intuitively rather obvious. Still, we cannot rely on our intuition when dealing with $\mathbf{R}^n$.)

61. Consider two subspaces V and W of $\mathbf{R}^n$, where V is contained in W. In Exercise 60, we learned that $\dim(V) \le \dim(W)$. Show that if $\dim(V) = \dim(W)$, then $V = W$.

62. Consider a subspace V of $\mathbf{R}^n$ with $\dim(V) = n$. Explain why $V = \mathbf{R}^n$.

81. Consider a 4×2 matrix **A** and a 2×5 matrix **B**.
a. What are the possible dimensions of the *kernel* of **AB**?
b. What are the possible dimensions of the *image* of **AB**?

Section 3.4:

In Exercises 5, 7, 17, and 18, determine whether the vector **x** is in the span V of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ (proceed "by inspection" if possible, and use the reduced row-echelon form if necessary). If **x** is in V, find the coordinates of **x** with respect to the basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ of V, and write the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$.

5. $\mathbf{x} = \begin{bmatrix} 7 \\ 16 \end{bmatrix};\ \mathbf{v}_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 5 \\ 12 \end{bmatrix}$

7. $\mathbf{x} = \begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix};\ \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

17. $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix};\ \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix},\ \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 0 \end{bmatrix},\ \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 4 \\ 1 \end{bmatrix}$

18. $\mathbf{x} = \begin{bmatrix} 5 \\ 4 \\ 3 \\ 2 \end{bmatrix};\ \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix},\ \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix},\ \mathbf{v}_3 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$

27. Find the matrix **B** of the linear transformation $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$ with respect to the basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ where

$$\mathbf{A} = \begin{bmatrix} 4 & 2 & -4 \\ 2 & 1 & -2 \\ -4 & -2 & 4 \end{bmatrix};\ \mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix};\ \mathbf{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix};\ \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be any basis of $\mathbf{R}^3$ consisting of perpendicular unit vectors, such that $\mathbf{v}_1 \times \mathbf{v}_2 = \mathbf{v}_3$.

32. Find the $\mathcal{B}$-matrix **B** of the linear transformation $T(\mathbf{x}) = \mathbf{x} \times \mathbf{v}_3$ from $\mathbf{R}^3$ to $\mathbf{R}^3$. Interpret T geometrically.

33. Find the $\mathcal{B}$-matrix **B** of the linear transformation $T(\mathbf{x}) = (\mathbf{v}_2 \cdot \mathbf{x})\mathbf{v}_2$ from $\mathbf{R}^3$ to $\mathbf{R}^3$. Interpret T geometrically.

34. Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be any basis of $\mathbf{R}^3$ consisting of perpendicular unit vectors, such that $\mathbf{v}_1 \times \mathbf{v}_2 = \mathbf{v}_3$. Find the $\mathcal{B}$-matrix **B** of the linear transformation $T(\mathbf{x}) = \mathbf{x} - 2(\mathbf{v}_3 \cdot \mathbf{x})\mathbf{v}_3$ from $\mathbf{R}^3$ to $\mathbf{R}^3$. Interpret T geometrically.

37. Find a basis $\mathcal{B}$ of $\mathbf{R}^n$ such that the $\mathcal{B}$ -matrix **B** of the linear transformation given by orthogonal projection T onto the line in $\mathbf{R}^2$ spanned by $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is diagonal.

39. Find a basis $\mathcal{B}$ of $\mathbf{R}^n$ such that the $\mathcal{B}$-matrix **B** of the linear transformation given by reflection T about the line in $\mathbf{R}^3$ spanned by $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is diagonal.

45. Consider the plane $2x_1 - 3x_2 + 4x_3 = 0$ Find a basis $\mathscr{B}$ of this plane such that $\mathbf{x}_{\mathscr{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ for $\mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$.

55. Consider the basis $\mathscr{B}$ of $\mathbf{R}^2$ consisting of the vectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, and let $\mathscr{R}$ be the basis consisting of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. Find a matrix **P** such that $[\mathbf{x}]_{\mathscr{R}} = \mathbf{P}[\mathbf{x}]_{\mathscr{B}}$.

Chapter 3 True/False

1. If $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are linearly independent vectors in $\mathbb{R}^n$, then they must form a basis of $\mathbb{R}^n$.

2. There exists a 5×4 matrix whose image consists of all of $\mathbb{R}^5$.

3. The kernel of any invertible matrix consists of the zero vector only.

4. The identity matrix I_n is similar to all invertible $n \times n$ matrices.

5. If $2\vec{u} + 3\vec{v} + 4\vec{w} = 5\vec{u} + 6\vec{v} + 7\vec{w}$, then vectors $\vec{u}, \vec{v}, \vec{w}$ must be linearly dependent.

6. The column vectors of a 5×4 matrix must be linearly dependent.

7. If $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ and $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m$ are any two bases of a subspace V of $\mathbb{R}^{10}$, then n must equal m.

8. If A is a 5×6 matrix of rank 4, then the nullity of A is 1.

9. The image of a 3×4 matrix is a subspace of $\mathbb{R}^4$.

10. The span of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ consists of all linear combinations of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$.

11. If vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ are linearly independent, then vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ must be linearly independent as well.

12. The vectors of the form $\begin{bmatrix} a \\ b \\ 0 \\ a \end{bmatrix}$ (where a and b are arbitrary real numbers) form a subspace of $\mathbb{R}^4$.

13. Matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ is similar to $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

14. Vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ form a basis of $\mathbb{R}^3$.

15. If the kernel of a matrix A consists of the zero vector only, then the column vectors of A must be linearly independent.

16. If the image of an $n \times n$ matrix A is all of $\mathbb{R}^n$, then A must be invertible.

17. If vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ span $\mathbb{R}^4$, then n must be equal to 4.

18. If vectors $\vec{u}, \vec{v}$, and $\vec{w}$ are in a subspace V of $\mathbb{R}^n$, then vector $2\vec{u} - 3\vec{v} + 4\vec{w}$ must be in V as well.

19. If matrix A is similar to matrix B, and B is similar to C, then C must be similar to A.

20. If a subspace V of $\mathbb{R}^n$ contains none of the standard vectors $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$, then V consists of the zero vector only.

21. If A and B are $n \times n$ matrices, and vector $\vec{v}$ is in the kernel of both A and B, then $\vec{v}$ must be in the kernel of matrix AB as well.

22. If two nonzero vectors are linearly dependent, then each of them is a scalar multiple of the other.

23. If $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are any three distinct vectors in $\mathbb{R}^3$, then there must be a linear transformation T from $\mathbb{R}^3$ to $\mathbb{R}^3$ such that $T(\vec{v}_1) = \vec{e}_1$, $T(\vec{v}_2) = \vec{e}_2$, and $T(\vec{v}_3) = \vec{e}_3$.

24. If vectors $\vec{u}, \vec{v}, \vec{w}$ are linearly dependent, then vector $\vec{w}$ must be a linear combination of $\vec{u}$ and $\vec{v}$.

25. If A and B are invertible $n \times n$ matrices, then AB must be similar to BA.

26. If A is an invertible $n \times n$ matrix, then the kernels of A and A^{-1} must be equal.

27. Matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is similar to $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

28. Vectors $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}, \begin{bmatrix} 9 \\ 8 \\ 7 \\ 6 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ -2 \end{bmatrix}$ are linearly independent.

29. If a subspace V of $\mathbb{R}^3$ contains the standard vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$, then V must be $\mathbb{R}^3$.

30. If a 2×2 matrix P represents the orthogonal projection onto a line in $\mathbb{R}^2$, then P must be similar to matrix $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

31. $\mathbb{R}^2$ is a subspace of $\mathbb{R}^3$.

32. If an $n \times n$ matrix A is similar to matrix B, then $A + 7I_n$ must be similar to $B + 7I_n$.

33. If V is any three-dimensional subspace of $\mathbb{R}^5$, then V has infinitely many bases.

34. Matrix I_n is similar to $2I_n$.

35. If $AB = 0$ for two 2×2 matrices A and B, then BA must be the zero matrix as well.

36. If A and B are $n \times n$ matrices, and vector $\vec{v}$ is in the image of both A and B, then $\vec{v}$ must be in the image of matrix $A + B$ as well.

37. If V and W are subspaces of $\mathbb{R}^n$, then their union $V \cup W$ must be a subspace of $\mathbb{R}^n$ as well.

38. If the kernel of a 5×4 matrix A consists of the zero vector only and if $A\vec{v} = A\vec{w}$ for two vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^4$, then vectors $\vec{v}$ and $\vec{w}$ must be equal.

39. If $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ and $\vec{w}_1, \vec{w}_2, \dots, \vec{w}_n$ are two bases of $\mathbb{R}^n$, then there exists a linear transformation T from $\mathbb{R}^n$ to $\mathbb{R}^n$ such that $T(\vec{v}_1) = \vec{w}_1$, $T(\vec{v}_2) = \vec{w}_2, \dots, T(\vec{v}_n) = \vec{w}_n$.

40. If matrix A represents a rotation through $\pi/2$ and matrix B a rotation through $\pi/4$, then A is similar to B.

41. There exists a 2×2 matrix A such that $\text{im}(A) = \ker(A)$.

42. If two $n \times n$ matrices A and B have the same rank, then they must be similar.

43. If A is similar to B, and A is invertible, then B must be invertible as well.

44. If $A^2 = 0$ for a 10×10 matrix A, then the inequality $\text{rank}(A) \le 5$ must hold.

45. For every subspace V of $\mathbb{R}^3$ there exists a 3×3 matrix A such that $V = \text{im}(A)$.

46. There exists a nonzero 2×2 matrix A that is similar to $2A$.

47. If the 2×2 matrix R represents the reflection about a line in $\mathbb{R}^2$, then R must be similar to matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

48. If A is similar to B, then there exists one *and only one* invertible matrix S such that $S^{-1}AS = B$.

49. If the kernel of a 5×4 matrix A consists of the zero vector alone, and if $AB = AC$ for two 4×5 matrices B and C, then matrices B and C must be equal.

50. If A is any $n \times n$ matrix such that $A^2 = A$, then the image of A and the kernel of A have only the zero vector in common.

51. There exists a 2×2 matrix A such that $A^2 \neq 0$ and $A^3 = 0$.

52. If A and B are $n \times m$ matrices such that the image of A is a subset of the image of B, then there must exist an $m \times m$ matrix C such that $A = BC$.

53. Among the 3×3 matrices whose entries are all 0's and 1's, most are invertible.