

Math E-21b – Spring 2025 – Homework #3

Section 2.3:

Problem 1. (2.3/14) For the matrices $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $\mathbf{B} = [1 \ 2 \ 3]$, $\mathbf{C} = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$, $\mathbf{D} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{E} = [5]$,

determine which of the 25 matrix products $\mathbf{AA}, \mathbf{AB}, \mathbf{AC}, \dots, \mathbf{ED}, \mathbf{EE}$ are defined, and compute those that are defined.

Section 2.4:

Problem 2. (2.4/66) Consider two $n \times n$ matrices \mathbf{A} and \mathbf{B} , such that the product \mathbf{AB} is invertible. Show that the matrices \mathbf{A} and \mathbf{B} are both invertible. *Hint:* $\mathbf{AB}(\mathbf{AB})^{-1} = \mathbf{I}_n$ and $(\mathbf{AB})^{-1}\mathbf{AB} = \mathbf{I}_n$. Use Fact 2.4.8.

Problem 3. (2.4/67-75) For two invertible $n \times n$ matrices \mathbf{A} and \mathbf{B} , determine which of the formulas stated below are necessarily true.

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|---|--|---|
| 67. $(\mathbf{A} + \mathbf{B})^2 = \mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2$ | 70. $(\mathbf{A} - \mathbf{B})(\mathbf{A} + \mathbf{B}) = \mathbf{A}^2 - \mathbf{B}^2$ | 73. $(\mathbf{ABA}^{-1})^3 = \mathbf{AB}^3\mathbf{A}^{-1}$ |
| 68. \mathbf{A}^2 is invertible, and $(\mathbf{A}^2)^{-1} = (\mathbf{A}^{-1})^2$ | 71. $\mathbf{ABB}^{-1}\mathbf{A}^{-1} = \mathbf{I}_n$ | 74. $(\mathbf{I}_n + \mathbf{A})(\mathbf{I}_n + \mathbf{A}^{-1}) = 2\mathbf{I}_n + \mathbf{A} + \mathbf{A}^{-1}$ |
| 69. $\mathbf{A} + \mathbf{B}$ is invertible, and $(\mathbf{A} + \mathbf{B})^{-1} = \mathbf{A}^{-1} + \mathbf{B}^{-1}$ | 72. $\mathbf{ABA}^{-1} = \mathbf{B}$ | 75. $\mathbf{A}^{-1}\mathbf{B}$ is invertible, and $(\mathbf{A}^{-1}\mathbf{B})^{-1} = \mathbf{B}^{-1}\mathbf{A}$ |

Problem 4. (2.4/76) Find all linear transformations T from \mathbf{R}^2 to \mathbf{R}^2 such that $T \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $T \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

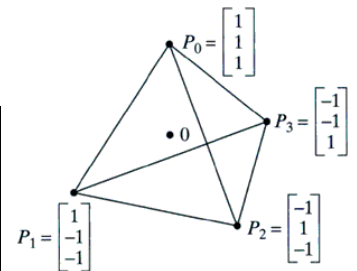
Hint: We are looking for the 2×2 matrices \mathbf{A} such that $\mathbf{A} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{A} \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

These two equations can be combined to form the matrix equation $\mathbf{A} \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$.

Problem 5. (2.4/78) Find the matrix \mathbf{A} of the linear transformation T from \mathbf{R}^2 to \mathbf{R}^3 with

$$T \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \\ 3 \end{bmatrix} \text{ and } T \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Problem 6. (2.4/80) Consider the regular tetrahedron sketched at right, whose center is at the origin.



Let T from \mathbf{R}^3 to \mathbf{R}^3 be the rotation about the axis through points 0 and P_2 that transforms P_1 into P_3 . Find the images of the four corners of the tetrahedron under this transformation.	$P_0 \xrightarrow{T}$	Let L from \mathbf{R}^3 to \mathbf{R}^3 be the reflection about the plane through the points $0, P_0$, and P_3 . Find the images of the four corners of the tetrahedron under this transformation.	$P_0 \xrightarrow{L}$
	$P_1 \rightarrow P_3$		$P_1 \rightarrow$
	$P_2 \rightarrow$		$P_2 \rightarrow$
	$P_3 \rightarrow$		$P_3 \rightarrow$

Describe the transformations in parts (a) through (c) geometrically.

a. T^{-1} b. L^{-1} c. $T^2 = T \circ T$ (the composite of T with itself)

d. Find the images of the four corners under the transformations $T \circ L$ and $L \circ T$. Are the two transformations the same?

e. Find the images of the four corners under the transformation $L \circ T \circ L$.

Describe this transformation geometrically.

- | | |
|-------------------------------|-------------------------------|
| $P_0 \xrightarrow{T \circ L}$ | $P_0 \xrightarrow{L \circ T}$ |
| $P_1 \rightarrow$ | $P_1 \rightarrow$ |
| $P_2 \rightarrow$ | $P_2 \rightarrow$ |
| $P_3 \rightarrow$ | $P_3 \rightarrow$ |

Problem 7. (2.4/81) Find the matrices of the transformations T and L defined in Problem 6 above.

Section 3.1:

For each matrix \mathbf{A} in Problems 8-10, find vectors that span the kernel of \mathbf{A} . Use paper and pencil.

Problem 8. (3.1/6) $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Problem 9. (3.1/8) $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

Problem 10. (3.1/12) $\mathbf{A} = \begin{bmatrix} 1 & -1 & -1 & 1 & 1 \\ -1 & 1 & 0 & -2 & 2 \\ 1 & -1 & -2 & 0 & 3 \\ 2 & -2 & -1 & 3 & 4 \end{bmatrix}$

For each matrix \mathbf{A} in Problems 11-12, describe the image of the transformation $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$ geometrically (as a line, plane, etc. in \mathbf{R}^2 or \mathbf{R}^3). [A brief justification is preferred.]

Problem 11. (3.1/20) $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Problem 12. (3.1/22) $\mathbf{A} = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 4 & 2 \\ 6 & 5 & 7 \end{bmatrix}$

Problem 13. (3.1/32) Give an example of a linear transformation whose image is the line spanned by $\begin{bmatrix} 7 \\ 6 \\ 5 \end{bmatrix}$ in \mathbf{R}^3 .

Problem 14. (3.1/34) Give an example of a linear transformation whose kernel is the line spanned by $\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$ in \mathbf{R}^3 .

Problem 15. (3.1/39) Consider an $n \times p$ matrix \mathbf{A} and a $p \times m$ matrix \mathbf{B} .

- What is the relationship between $\ker(\mathbf{AB})$ and $\ker(\mathbf{B})$? Are they always equal? Is one of them always contained in the other?
- What is the relationship between $\text{im}(\mathbf{A})$ and $\text{im}(\mathbf{AB})$?

Problem 16. (3.1/44) Consider a matrix \mathbf{A} , and let $\mathbf{B} = \text{rref}(\mathbf{A})$.

- Is $\ker(\mathbf{A})$ necessarily equal to $\ker(\mathbf{B})$? Explain.
- Is $\text{im}(\mathbf{A})$ necessarily equal to $\text{im}(\mathbf{B})$? Explain.

For additional practice (not to be turned in):

Section 2.3:

If possible, compute the matrix products in Exercises 1, 2, 3, 4, 7, 10, 11, and 12, using pencil and paper.

1. $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ 2. $\begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 7 & 5 \\ 3 & 1 \end{bmatrix}$ 3. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ 4. $\begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$

7. $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$ 10. $\begin{bmatrix} 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 1 \end{bmatrix}$ 11. $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ 12. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

27. Prove the *distributive laws* for matrices: $\mathbf{A}(\mathbf{C} + \mathbf{D}) = \mathbf{AC} + \mathbf{AD}$ and $(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$.

Section 3.1:

For each matrix \mathbf{A} in Exercises 1, 2, and 5, find vectors that span the kernel of \mathbf{A} . Use paper and pencil.

1. $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ 2. $\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix}$ 5. $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{bmatrix}$

19. For the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & -4 & -6 & -8 \end{bmatrix}$, describe the image of the transformation $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$ geometrically (as a line, plane, etc. in \mathbf{R}^2 or \mathbf{R}^3).

Describe the images and kernels of the transformations in Exercises 23 through 25 geometrically.

23. Reflection about the line $y = \frac{1}{3}x$ in \mathbf{R}^2 .
 24. Orthogonal projection onto the plane $x + 2y + 3z = 0$ in \mathbf{R}^3 .
 25. Rotation through an angle $\frac{\pi}{4}$ in the counterclockwise direction (in \mathbf{R}^2).

31. Give an example of a matrix \mathbf{A} such that $\text{im}(\mathbf{A})$ is the plane with normal vector $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ in \mathbf{R}^3 .

Extra problems for those interested in economics (not to be turned in):

Section 2.4:

101. Consider two $n \times n$ matrices \mathbf{A} and \mathbf{B} whose entries are positive or zero. Suppose that all entries of \mathbf{A} are less than or equal to s , and all column sums of \mathbf{B} are less than or equal to r (the j th column sum of a matrix is the sum of all the entries in the j th column). Show that all entries of the matrix \mathbf{AB} are less than or equal to sr .

102. (This exercise builds on Exercise 101.) Consider an $n \times n$ matrix \mathbf{A} whose entries are positive or zero. Suppose that all column sums of \mathbf{A} are less than 1. Let r be the largest column sum of \mathbf{A} .

- Show that the entries of \mathbf{A}^m are less than or equal to r^m , for all positive integers m .
- Show that $\lim_{m \rightarrow \infty} \mathbf{A}^m = \mathbf{0}$ (meaning that all entries of \mathbf{A}^m approach zero).
- Show that the infinite series $\mathbf{I}_n + \mathbf{A} + \mathbf{A}^2 + \dots + \mathbf{A}^m + \dots$ converges (entry by entry).
- Compute the product $(\mathbf{I}_n - \mathbf{A})(\mathbf{I}_n + \mathbf{A} + \mathbf{A}^2 + \dots + \mathbf{A}^m)$. Simplify the result. Then let m go to infinity, and thus show that $(\mathbf{I}_n - \mathbf{A})^{-1} = \mathbf{I}_n + \mathbf{A} + \mathbf{A}^2 + \dots + \mathbf{A}^m + \dots$.

103. (This exercise builds on Exercises 49, 101, and 102.)

- Consider the industries J_1, J_2, \dots, J_n in an economy. We say that the industry J_j is *productive* if the j th column sum of the technology matrix \mathbf{A} is less than 1. What does this mean in terms of economics?
- We say that an economy is productive if all of its industries are productive. Exercise 102 shows that if \mathbf{A} is the technology matrix of a productive economy, then the matrix $(\mathbf{I}_n - \mathbf{A})$ is invertible. What does this result tell you about the ability of a productive economy to satisfy consumer demand?
- Interpret the formula $(\mathbf{I}_n - \mathbf{A})^{-1} = \mathbf{I}_n + \mathbf{A} + \mathbf{A}^2 + \dots + \mathbf{A}^m + \dots$ derived in Exercise 102(d) in terms of economics.

Chapter 2 True/False questions (for additional practice – not to be turned in)

- If A is any invertible $n \times n$ matrix, then $\text{ref}(A) = I_n$.
- The formula $(A^2)^{-1} = (A^{-1})^2$ holds for all invertible matrices A .
- The formula $AB = BA$ holds for all $n \times n$ matrices A and B .
- If $AB = I_n$ for two $n \times n$ matrices A and B , then A must be the inverse of B .
- If A is a 3×4 matrix and B is a 4×5 matrix, then AB will be a 5×3 matrix.
- The function $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ 1 \end{bmatrix}$ is a linear transformation.
- The matrix $\begin{bmatrix} 5 & 6 \\ -6 & 5 \end{bmatrix}$ represents a rotation combined with a scaling.
- If A is any invertible $n \times n$ matrix, then A commutes with A^{-1} .
- The function $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x - y \\ y - x \end{bmatrix}$ is a linear transformation.

10. Matrix $\begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$ represents a rotation.
11. There exists a real number k such that the matrix $\begin{bmatrix} k-2 & 3 \\ -3 & k-2 \end{bmatrix}$ fails to be invertible.
12. Matrix $\begin{bmatrix} -0.6 & 0.8 \\ -0.8 & -0.6 \end{bmatrix}$ represents a rotation.
13. The formula $\det(2A) = 2\det(A)$ holds for all 2×2 matrices A .
14. There exists a matrix A such that $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} A \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.
15. Matrix $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ is invertible.
16. Matrix $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ is invertible.
17. There exists an upper triangular 2×2 matrix A such that $A^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.
18. The function $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (y+1)^2 - (y-1)^2 \\ (x-3)^2 - (x+3)^2 \end{bmatrix}$ is a linear transformation.
19. Matrix $\begin{bmatrix} k & -2 \\ 5 & k-6 \end{bmatrix}$ is invertible for all real numbers k .
20. There exists a real number k such that the matrix $\begin{bmatrix} k-1 & -2 \\ -4 & k-3 \end{bmatrix}$ fails to be invertible.
21. The matrix product $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ is always a scalar multiple of I_2 .
22. There exists a nonzero upper triangular 2×2 matrix A such that $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.
23. There exists a positive integer n such that $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^n = I_2$.
24. There exists an invertible 2×2 matrix A such that $A^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.
25. There exists an invertible $n \times n$ matrix with two identical rows.
26. If $A^2 = I_n$, then matrix A must be invertible.
27. There exists a matrix A such that $A \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$.
28. There exists a matrix A such that $\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.
29. The matrix $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ represents a reflection about a line.
30. $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}^3 = \begin{bmatrix} 1 & 3k \\ 0 & 1 \end{bmatrix}$ for all real numbers k .
31. If matrix $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ is invertible, then matrix $\begin{bmatrix} a & b \\ d & e \end{bmatrix}$ must be invertible as well.
32. If A^2 is invertible, then matrix A itself must be invertible.
33. If $A^{17} = I_2$, then matrix A must be I_2 .
34. If $A^2 = I_2$, then matrix A must be either I_2 or $-I_2$.
35. If matrix A is invertible, then matrix $5A$ must be invertible as well.
36. If A and B are two 4×3 matrices such that $A\vec{v} = B\vec{v}$ for all vectors \vec{v} in \mathbb{R}^3 , then matrices A and B must be equal.
37. If matrices A and B commute, then the formula $A^2B = BA^2$ must hold.
38. If $A^2 = A$ for an invertible $n \times n$ matrix A , then A must be I_n .
39. If matrices A and B are both invertible, then matrix $A + B$ must be invertible as well.
40. The equation $A^2 = A$ holds for all 2×2 matrices A representing a projection.
41. The equation $A^{-1} = A$ holds for all 2×2 matrices A representing a reflection.
42. The formula $(A\vec{v}) \cdot (A\vec{w}) = \vec{v} \cdot \vec{w}$ holds for all invertible 2×2 matrices A and for all vectors \vec{v} and \vec{w} in \mathbb{R}^2 .
43. There exist a 2×3 matrix A and a 3×2 matrix B such that $AB = I_2$.
44. There exist a 3×2 matrix A and a 2×3 matrix B such that $AB = I_3$.
45. If $A^2 + 3A + 4I_3 = 0$ for a 3×3 matrix A , then A must be invertible.
46. If A is an $n \times n$ matrix such that $A^2 = 0$, then matrix $I_n + A$ must be invertible.
47. If matrix A commutes with B , and B commutes with C , then matrix A must commute with C .
48. If T is any linear transformation from \mathbb{R}^3 to \mathbb{R}^3 , then $T(\vec{v} \times \vec{w}) = T(\vec{v}) \times T(\vec{w})$ for all vectors \vec{v} and \vec{w} in \mathbb{R}^3 .
49. There exists an invertible 10×10 matrix that has 92 ones among its entries.

50. The formula $\text{rref}(AB) = \text{rref}(A) \text{rref}(B)$ holds for all $n \times p$ matrices A and for all $p \times m$ matrices B .
51. There exists an invertible matrix S such that $S^{-1} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} S$ is a diagonal matrix.
52. If the linear system $A^2 \vec{x} = \vec{b}$ is consistent, then the system $A \vec{x} = \vec{b}$ must be consistent as well.
53. There exists an invertible 2×2 matrix A such that $A^{-1} = -A$.
54. There exists an invertible 2×2 matrix A such that $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.
55. If a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ represents the orthogonal projection onto a line L , then the equation $a^2 + b^2 + c^2 + d^2 = 1$ must hold.
56. If A is an invertible 2×2 matrix and B is any 2×2 matrix, then the formula $\text{rref}(AB) = \text{rref}(B)$ must hold.