Math E-21b – Spring 2024 – Homework #3

Section 2.3:

Problem 1. (2.3/14) For the matrices $\mathbf{A}=\begin{bmatrix}1&1\\1&1\end{bmatrix}$, $\mathbf{B}=\begin{bmatrix}1&2&3\end{bmatrix}$, $\mathbf{C}=\begin{bmatrix}1&0&-1\\2&1&0\\3&2&1\end{bmatrix}$, $\mathbf{D}=\begin{bmatrix}1\\1\\1\end{bmatrix}$, $\mathbf{E}=\begin{bmatrix}5\end{bmatrix}$,

determine which of the 25 matrix products **AA**, **AB**, **AC**, …, **ED**, **EE** are defined, and compute those that are defined.

Section 2.4:

Problem 2. (2.4/66) Consider two $n\times n$ matrices **A** and **B**, such that the product **AB** is invertible. Show that the matrices **A** and **B** are both invertible. *Hint*: $\mathbf{AB}(\mathbf{AB})^{-1}=\mathbf{I}_n$ and $(\mathbf{AB})^{-1}\mathbf{AB}=\mathbf{I}_n$. Use Fact 2.4.8.

Problem 3. (2.4/67-75) For two invertible $n\times n$ matrices **A** and **B**, determine which of the formulas stated below are necessarily true.

67. $(\mathbf{A}+\mathbf{B})^2=\mathbf{A}^2+2\mathbf{AB}+\mathbf{B}^2$
68. $\mathbf{A}^2$ is invertible, and $(\mathbf{A}^2)^{-1}=(\mathbf{A}^{-1})^2$
69. $\mathbf{A}+\mathbf{B}$ is invertible, and $(\mathbf{A}+\mathbf{B})^{-1}=\mathbf{A}^{-1}+\mathbf{B}^{-1}$
70. $(\mathbf{A}-\mathbf{B})(\mathbf{A}+\mathbf{B})=\mathbf{A}^2-\mathbf{B}^2$
71. $\mathbf{ABB}^{-1}\mathbf{A}^{-1}=\mathbf{I}_n$
72. $\mathbf{ABA}^{-1}=\mathbf{B}$
73. $(\mathbf{ABA}^{-1})^3=\mathbf{AB}^3\mathbf{A}^{-1}$
74. $(\mathbf{I}_n+\mathbf{A})(\mathbf{I}_n+\mathbf{A}^{-1})=2\mathbf{I}_n+\mathbf{A}+\mathbf{A}^{-1}$
75. $\mathbf{A}^{-1}\mathbf{B}$ is invertible, and $(\mathbf{A}^{-1}\mathbf{B})^{-1}=\mathbf{B}^{-1}\mathbf{A}$

Problem 4. (2.4/76) Find all linear transformations T from $\mathbf{R}^2$ to $\mathbf{R}^2$ such that $T\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}2\\1\end{bmatrix}$ and $T\begin{bmatrix}2\\5\end{bmatrix}=\begin{bmatrix}1\\3\end{bmatrix}$.

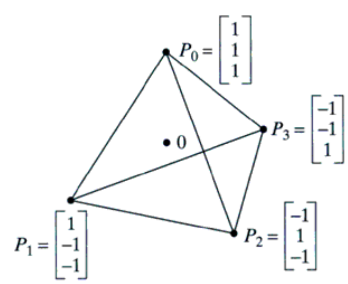
Hint: We are looking for the 2×2 matrices **A** such that $\mathbf{A}\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}2\\1\end{bmatrix}$ and $\mathbf{A}\begin{bmatrix}2\\5\end{bmatrix}=\begin{bmatrix}1\\3\end{bmatrix}$.

These two equations can be combined to form the matrix equation $\mathbf{A}\begin{bmatrix}1&2\\2&5\end{bmatrix}=\begin{bmatrix}2&1\\1&3\end{bmatrix}$.

Problem 5. (2.4/78) Find the matrix **A** of the linear transformation T from $\mathbf{R}^2$ to $\mathbf{R}^3$ with

$$T\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}7\\5\\3\end{bmatrix} \text{ and } T\begin{bmatrix}2\\5\end{bmatrix}=\begin{bmatrix}1\\2\\3\end{bmatrix}.$$

Problem 6. (2.4/80) Consider the regular tetrahedron sketched at right, whose center is at the origin.



Let T from $\mathbf{R}^3$ to $\mathbf{R}^3$ be the rotation about the axis through points 0 and P_2 that transforms P_1 into P_3. Find the images of the four corners of the tetrahedron under this transformation.	$P_0\xrightarrow{T}$ $P_1\to P_3$ $P_2\to$ $P_3\to$	Let L from $\mathbf{R}^3$ to $\mathbf{R}^3$ be the reflection about the plane through the points 0, P_0, and P_3. Find the images of the four corners of the tetrahedron under this transformation.	$P_0\xrightarrow{L}$ $P_1\to$ $P_2\to$ $P_3\to$

Describe the transformations in parts (a) through (c) geometrically.

a. T^{-1} b. L^{-1} c. $T^2=T\circ T$ (the composite of T with itself)

d. Find the images of the four corners under the transformations $T\circ L$ and $L\circ T$. Are the two transformations the same?

e. Find the images of the four corners under the transformation $L\circ T\circ L$. Describe this transformation geometrically.

$P_0\xrightarrow{T\circ L}$	$P_0\xrightarrow{L\circ T}$
$P_1\to$	$P_1\to$
$P_2\to$	$P_2\to$
$P_3\to$	$P_3\to$

Problem 7. (2.4/81) Find the matrices of the transformations T and L defined in Problem 6 above.

Section 3.1:

For each matrix **A** in Problems 8-10, find vectors that span the kernel of **A**. Use paper and pencil.

Problem 8. (3.1/6) $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Problem 9. (3.1/8) $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

Problem 10. (3.1/12) $\mathbf{A} = \begin{bmatrix} 1 & -1 & -1 & 1 & 1 \\ -1 & 1 & 0 & -2 & 2 \\ 1 & -1 & -2 & 0 & 3 \\ 2 & -2 & -1 & 3 & 4 \end{bmatrix}$

For each matrix **A** in Problems 11-12, describe the image of the transformation $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$ geometrically (as a line, plane, etc. in $\mathbf{R}^2$ or $\mathbf{R}^3$). [A brief justification is preferred.]

Problem 11. (3.1/20) $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Problem 12. (3.1/22) $\mathbf{A} = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 4 & 2 \\ 6 & 5 & 7 \end{bmatrix}$

Problem 13. (3.1/32) Give an example of a linear transformation whose image is the line spanned by $\begin{bmatrix} 7 \\ 6 \\ 5 \end{bmatrix}$ in $\mathbf{R}^3$.

Problem 14. (3.1/34) Give an example of a linear transformation whose kernel is the line spanned by $\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$ in $\mathbf{R}^3$.

Problem 15. (3.1/39) Consider an $n \times p$ matrix **A** and a $p \times m$ matrix **B**.

a. What is the relationship between ker(**AB**) and ker(**B**)? Are they always equal? Is one of them always contained in the other?

b. What is the relationship between im(**A**) and im(**AB**)?

Problem 16. (3.1/44) Consider a matrix **A**, and let **B** = rref(**A**).

a. Is ker(**A**) necessarily equal to ker(**B**)? Explain.

b. Is im(**A**) necessarily equal to im(**B**)? Explain.

For additional practice (not to be turned in):

Section 2.3:

If possible, compute the matrix products in Exercises 1, 2, 3, 4, 7, 10, 11, and 12, using pencil and paper.

1. $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

2. $\begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 7 & 5 \\ 3 & 1 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

4. $\begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$

7. $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & -2 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$

10. $\begin{bmatrix} 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 1 \end{bmatrix}$

11. $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

12. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

27. Prove the *distributive laws* for matrices: $\mathbf{A}(\mathbf{C}+\mathbf{D}) = \mathbf{AC}+\mathbf{AD}$ and $(\mathbf{A}+\mathbf{B})\mathbf{C} = \mathbf{AC}+\mathbf{BC}$.

Section 3.1:

For each matrix **A** in Exercises 1, 2, and 5, find vectors that span the kernel of **A**. Use paper and pencil.

1. $\mathbf{A}=\begin{bmatrix}1&2\\3&4\end{bmatrix}$ 2. $\mathbf{A}=\begin{bmatrix}2&3\\6&9\end{bmatrix}$ 5. $\mathbf{A}=\begin{bmatrix}1&1&1\\1&2&3\\1&3&5\end{bmatrix}$

19. For the matrix $\mathbf{A}=\begin{bmatrix}1&2&3&4\\-2&-4&-6&-8\end{bmatrix}$, describe the image of the transformation $T(\mathbf{x})=\mathbf{Ax}$ geometrically (as a line, plane, etc. in $\mathbf{R}^2$ or $\mathbf{R}^3$).

Describe the images and kernels of the transformations in Exercises 23 through 25 geometrically.

23. Reflection about the line $y=\frac{1}{3}x$ in $\mathbf{R}^2$.

24. Orthogonal projection onto the plane $x+2y+3z=0$ in $\mathbf{R}^3$.

25. Rotation through an angle $\pi/4$ in the counterclockwise direction (in $\mathbf{R}^2$).

31. Give an example of a matrix **A** such that im(**A**) is the plane with normal vector $\begin{bmatrix}1\\3\\2\end{bmatrix}$ in $\mathbf{R}^3$**.**

Extra problems for those interested in economics (not to be turned in):

<u>Section 2.4</u>:

101. Consider two $n \times n$ matrices **A** and **B** whose entries are positive or zero. Suppose that all entries of **A** are less than or equal to s, and all column sums of **B** are less than or equal to r (the jth column sum of a matrix is the sum of all the entries in the jth column). Show that all entries of the matrix **AB** are less than or equal to sr.

102. (This exercise builds on Exercise 101.) Consider an $n \times n$ matrix **A** whose entries are positive or zero. Suppose that all column sums of **A** are less 1. Let r be the largest column sum of **A**.

a. Show that the entries of $\mathbf{A}^m$ are less than or equal to r^m, for all positive integers m.

b. Show that $\lim_{m\to\infty}\mathbf{A}^m=\mathbf{0}$ (meaning that all entries of $\mathbf{A}^m$ approach zero).

c. Show that the infinite series $\mathbf{I}_n+\mathbf{A}+\mathbf{A}^2+\cdots+\mathbf{A}^m+\cdots$ converges (entry by entry).

d. Compute the product $(\mathbf{I}_n-\mathbf{A})(\mathbf{I}_n+\mathbf{A}+\mathbf{A}^2+\cdots+\mathbf{A}^m)$. Simplify the result. Then let m go to infinity, and thus show that $(\mathbf{I}_n-\mathbf{A})^{-1}=\mathbf{I}_n+\mathbf{A}+\mathbf{A}^2+\cdots+\mathbf{A}^m+\cdots$.

103. (This exercise builds on Exercises 49, 101, and 102.)

a. Consider the industries J_1, J_2, ..., J_n in an economy. We say that the industry J_j is *productive* if the jth column sum of the technology matrix **A** is less than 1. What does this mean in terms of economics?

b. We say that an economy is productive if all of its industries are productive. Exercise 102 shows that if **A** is the technology matrix of a productive economy, then the matrix ($\mathbf{I}_n$ – **A**) is invertible. What does this result tell you about the ability of a productive economy to satisfy consumer demand?

c. Interpret the formula $(\mathbf{I}_n-\mathbf{A})^{-1}=\mathbf{I}_n+\mathbf{A}+\mathbf{A}^2+\cdots+\mathbf{A}^m+\cdots$ derived in Exercise 102(d) in terms of economics.

Chapter 2 True/False questions (for additional practice – not to be turned in)

1. If A is any invertible $n \times n$ matrix, then $\mathrm{rref}(A) = I_n$.

2. The formula $(A^2)^{-1} = (A^{-1})^2$ holds for all invertible matrices A.

3. The formula $AB = BA$ holds for all $n \times n$ matrices A and B.

4. If $AB = I_n$ for two $n \times n$ matrices A and B, then A must be the inverse of B.

5. If A is a 3×4 matrix and B is a 4×5 matrix, then AB will be a 5×3 matrix.

10. Matrix $\begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$ represents a rotation.

11. There exists a real number k such that the matrix $\begin{bmatrix} k-2 & 3 \\ -3 & k-2 \end{bmatrix}$ fails to be invertible.

12. Matrix $\begin{bmatrix} -0.6 & 0.8 \\ -0.8 & -0.6 \end{bmatrix}$ represents a rotation.

13. The formula $\det(2A) = 2\det(A)$ holds for all 2×2 matrices A.

14. There exists a matrix A such that $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} A \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

6. The function $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ 1 \end{bmatrix}$ is a linear transformation.

7. The matrix $\begin{bmatrix} 5 & 6 \\ -6 & 5 \end{bmatrix}$ represents a rotation combined with a scaling.

8. If A is any invertible $n \times n$ matrix, then A commutes with A^{-1}.

9. The function $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x-y \\ y-x \end{bmatrix}$ is a linear transformation.

28. There exists a matrix A such that $\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

29. The matrix $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ represents a reflection about a line.

30. $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}^3 = \begin{bmatrix} 1 & 3k \\ 0 & 1 \end{bmatrix}$ for all real numbers k.

31. If matrix $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ is invertible, then matrix $\begin{bmatrix} a & b \\ d & e \end{bmatrix}$ must be invertible as well.

32. If A^2 is invertible, then matrix A itself must be invertible.

33. If $A^{17} = I_2$, then matrix A must be I_2.

34. If $A^2 = I_2$, then matrix A must be either I_2 or $-I_2$.

35. If matrix A is invertible, then matrix $5A$ must be invertible as well.

15. Matrix $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ is invertible.

16. Matrix $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ is invertible.

17. There exists an upper triangular 2×2 matrix A such that $A^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

18. The function $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (y+1)^2 - (y-1)^2 \\ (x-3)^2 - (x+3)^2 \end{bmatrix}$ is a linear transformation.

19. Matrix $\begin{bmatrix} k & -2 \\ 5 & k-6 \end{bmatrix}$ is invertible for all real numbers k.

20. There exists a real number k such that the matrix $\begin{bmatrix} k-1 & -2 \\ -4 & k-3 \end{bmatrix}$ fails to be invertible.

21. The matrix product $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ is always a scalar multiple of I_2.

22. There exists a nonzero upper triangular 2×2 matrix A such that $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

23. There exists a positive integer n such that $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^n = I_2$.

24. There exists an invertible 2×2 matrix A such that $A^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

25. There exists an invertible $n \times n$ matrix with two identical rows.

26. If $A^2 = I_n$, then matrix A must be invertible.

27. There exists a matrix A such that $A\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$.

36. If A and B are two 4×3 matrices such that $A\vec{v} = B\vec{v}$ for all vectors $\vec{v}$ in $\mathbb{R}^3$, then matrices A and B must be equal.

37. If matrices A and B commute, then the formula $A^2B = BA^2$ must hold.

38. If $A^2 = A$ for an invertible $n \times n$ matrix A, then A must be I_n.

39. If matrices A and B are both invertible, then matrix $A + B$ must be invertible as well.

40. The equation $A^2 = A$ holds for all 2×2 matrices A representing a projection.

41. The equation $A^{-1} = A$ holds for all 2×2 matrices A representing a reflection.

42. The formula $(A\vec{v}) \cdot (A\vec{w}) = \vec{v} \cdot \vec{w}$ holds for all invertible 2×2 matrices A and for all vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^2$.

43. There exist a 2×3 matrix A and a 3×2 matrix B such that $AB = I_2$.

44. There exist a 3×2 matrix A and a 2×3 matrix B such that $AB = I_3$.

45. If $A^2 + 3A + 4I_3 = 0$ for a 3×3 matrix A, then A must be invertible.

46. If A is an $n \times n$ matrix such that $A^2 = 0$, then matrix $I_n + A$ must be invertible.

47. If matrix A commutes with B, and B commutes with C, then matrix A must commute with C.

48. If T is any linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$, then $T(\vec{v} \times \vec{w}) = T(\vec{v}) \times T(\vec{w})$ for all vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^3$.

49. There exists an invertible 10×10 matrix that has 92 ones among its entries.

50. The formula $\text{rref}(AB) = \text{rref}(A)\,\text{rref}(B)$ holds for all $n \times p$ matrices A and for all $p \times m$ matrices B.

51. There exists an invertible matrix S such that $S^{-1}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}S$ is a diagonal matrix.

52. If the linear system $A^2\vec{x} = \vec{b}$ is consistent, then the system $A\vec{x} = \vec{b}$ must be consistent as well.

53. There exists an invertible 2×2 matrix A such that $A^{-1} = -A$.

54. There exists an invertible 2×2 matrix A such that $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

55. If a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ represents the orthogonal projection onto a line L, then the equation $a^2 + b^2 + c^2 + d^2 = 1$ must hold.

56. If A is an invertible 2×2 matrix and B is any 2×2 matrix, then the formula $\text{rref}(AB) = \text{rref}(B)$ must hold.