Problems due in Canvas no later than Sat, Feb 15, 8:00pm:

Section 2.1:

Problem 1. (2.1/8) Find the inverse of the linear transformation $\begin{cases} y_1 = x_1 + 7x_2 \\ y_2 = 3x_1 + 20x_2 \end{cases}$. [That is, solve for x_1, x_2 in terms of y_1, y_2.]

Problem 2. (2.1/24-30) Consider the circular face in the accompanying figure. For each of the matrices **A** in Exercises 24 through 30, draw a sketch showing the effect of the linear transformation $T(\mathbf{x}) = \mathbf{Ax}$ on this face.
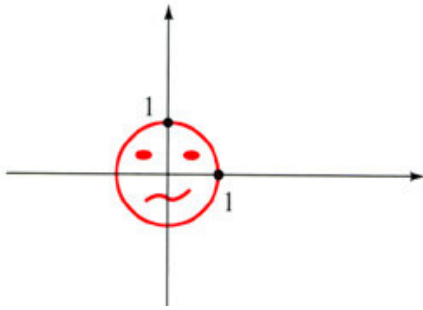
24. $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ 25. $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ 26. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ 27. $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

28. $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ 29. $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ 30. $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Problem 3. (2.1/44) The cross product of two vectors in $\mathbf{R}^3$ is defined by $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}$.

Consider an arbitrary (fixed) vector $\mathbf{v}$ in $\mathbf{R}^3$. Is the transformation $T(\mathbf{x}) = \mathbf{v} \times \mathbf{x}$ from $\mathbf{R}^3$ to $\mathbf{R}^3$ linear? If so, find its matrix in terms of the components of the vector $\mathbf{v}$.

Section 2.2:

Problem 4. (2.2/6) Let L be the line in $\mathbf{R}^3$ that consists of all scalar multiples of the vector $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$.

Find the orthogonal projection of the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ onto L.

Problem 5. (2.2/7) Let L be the line in $\mathbf{R}^3$ that consists of all scalar multiples of the vector $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$. Find the reflection of the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ about the line L.

Problem 6. (2.2/10) Find the matrix of the projection onto the line L in $\mathbf{R}^2$ shown in the accompanying figure. [*Note*: Your answer should be a 2×2 matrix.]

Problem 7. (2.2/11) Refer to Problem 6. Find the matrix of the reflection about the line L. [*Note*: As in the previous problem, your answer should be a 2×2 matrix.]
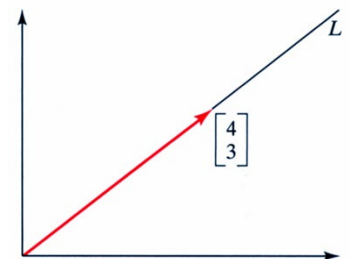
Find matrices of the linear transformations from $\mathbf{R}^3$ to $\mathbf{R}^3$ given in Problems 8-11. Some of these transformations have not been formally defined in the text. Use common sense. You may assume that all these transformations are linear. [*Note*: Your answers to each of these problems should be a 3×3 matrix.]

Problem 8. (2.2/19) The orthogonal projection onto the xy-plane.

Problem 9. (2.2/20) The reflection about the xz-plane.

Problem 10. (2.2/22) The rotation about the *y*-axis through an angle θ, counterclockwise as viewed from the positive *y*-axis.

Problem 11. (2.2/23) The reflection about the plane $y = z$.

Problem 12. (2.2/34) One of the five given matrices represents an orthogonal projection onto a line and another represents a reflection about a line. Identify both and briefly justify your choice.

$$\mathbf{A} = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \quad \mathbf{B} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \mathbf{C} = \frac{1}{3}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad \mathbf{D} = -\frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \quad \mathbf{E} = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$$

[Note: There are many different ways to solve this – some easy, some not so easy!]

Section 2.4:
Decide whether the matrices in Problems 13-15 are invertible. If they are, find the inverse matrix. Do the computations with paper and pencil. Show all your work.

Problem 13. (2.4/2) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$
     Problem 14. (2.4/4) $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix}$
     Problem 15. (2.4/6) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

Problem 16. (2.4/20) Decide whether the linear transformation $\begin{cases} y_1 = x_1 + 3x_2 + 3x_3 \\ y_2 = x_1 + 4x_2 + 8x_3 \\ y_3 = 2x_1 + 7x_2 + 12x_3 \end{cases}$ is invertible. Find the

inverse transformation if it exists. Do the computations with paper and pencil. Show all your work.

Problem 17. (2.4/54) Let $\mathbf{A} = \begin{bmatrix} 1 & 10 \\ -3 & 12 \end{bmatrix}$. Find a scalar λ and a nonzero vector $\mathbf{x}$ such that $\mathbf{Ax} = \lambda\mathbf{x}$.

[Note: You may wish to use Exercise 2.4/53 (see below) as a guide, but there are other approaches.]

For additional practice (<u>not to be turned in</u>):

Section 2.1:

5. Consider the linear transformation T from $\mathbf{R}^3$ to $\mathbf{R}^2$ with $T\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \end{bmatrix}$, $T\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \end{bmatrix}$, and $T\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -13 \\ 17 \end{bmatrix}$.

Find the matrix $\mathbf{A}$ of T.

7. Suppose $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are arbitrary vectors in $\mathbf{R}^n$. Consider the transformation from $\mathbf{R}^m$ to $\mathbf{R}^n$ given by

$$T\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_m\mathbf{v}_m.$$

Is this transformation linear? If so, find its matrix $\mathbf{A}$ in terms of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$.

9. Decide whether the matrix $\begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix}$ is invertible. Find the inverse if it exists.

37. Consider a linear transformation T from $\mathbf{R}^2$ to $\mathbf{R}^2$. Suppose that $\mathbf{v}$ and $\mathbf{w}$ are two arbitrary vectors in $\mathbf{R}^2$ and that $\mathbf{x}$ is a third vector whose endpoint is on the line segment connecting the endpoints of $\mathbf{v}$ and $\mathbf{w}$. Is the endpoint of the vector $T(\mathbf{x})$ necessarily on the line segment connecting the endpoints of $T(\mathbf{v})$ and $T(\mathbf{w})$?
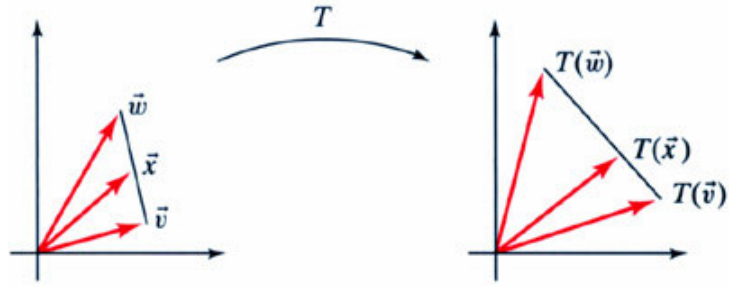
[Hint: We can write $\mathbf{x} = \mathbf{v} + k(\mathbf{w} - \mathbf{v})$, for some scalar k between 0 and 1.]

We can summarize this exercise by saying that a linear transformation maps a line onto a line.

43. a. Consider the vector $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$.

   Is the transformation $T(\mathbf{x}) = \mathbf{v} \cdot \mathbf{x}$ (the dot product) from $\mathbf{R}^3$ to $\mathbf{R}$ linear? If so, find the matrix of T.

   b. Consider an arbitrary vector $\mathbf{v}$ in $\mathbf{R}^3$. Is the transformation $T(\mathbf{x}) = \mathbf{v} \cdot \mathbf{x}$ linear?
   If so, find the matrix of T (in terms of the components of $\mathbf{v}$.

   c. Conversely, consider a linear transformation T from $\mathbf{R}^3$ to $\mathbf{R}$.
   Show that there exists a vector $\mathbf{v}$ in $\mathbf{R}^3$ such that $T(\mathbf{x}) = \mathbf{v} \cdot \mathbf{x}$, for all $\mathbf{x}$ in $\mathbf{R}^3$.

Section 2.2:

4. Interpret the following linear transformation geometrically: $T(\mathbf{x}) = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \mathbf{x}$.

5. The matrix $\begin{bmatrix} -0.8 & -0.6 \\ 0.6 & -0.8 \end{bmatrix}$ represents a rotation. Find the angle of rotation (in radians).

17. Consider a matrix A of the form $\mathbf{A} = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$, where $a^2 + b^2 = 1$. Find two nonzero perpendicular vectors

   $\mathbf{v}$ and $\mathbf{w}$ such that $\mathbf{A}\mathbf{v} = \mathbf{v}$ and $\mathbf{A}\mathbf{w} = -\mathbf{w}$ (write the entries of $\mathbf{v}$ and $\mathbf{w}$ in terms of a and b). Conclude that $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$ represents the reflection about the line L spanned by $\mathbf{v}$.

21. Find a matrix for the linear transformations from $\mathbf{R}^3$ to $\mathbf{R}^3$ that is rotation about the z-axis through an angle of $\pi/2$, counterclockwise as viewed from the positive z-axis.

24. Rotations and reflections have two remarkable properties: They preserve the length of vectors and the angle between vectors. (Draw figures illustrating these properties.) We will show that, conversely, any linear transformation T from $\mathbf{R}^2$ to $\mathbf{R}^2$ that preserves length and angles is either a rotation or a reflection. (about a line).

   a. Show that if $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$ preserves length and angles, then the two column vectors $\mathbf{v}$ and $\mathbf{w}$ of $\mathbf{A}$ must be perpendicular unit vectors.

   b. Write the first column vector of $\mathbf{A}$ as $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$.

   c. Show that if a linear transformation T from $\mathbf{R}^2$ to $\mathbf{R}^2$ preserves length and angles, then T is either a rotation or a reflection (about a line). See Exercise 17.

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27. Consider the matrices **A** through **E** below.

$$\mathbf{A} = \begin{bmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 0.36 & -0.48 \\ -0.48 & 0.64 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} -0.8 & 0.6 \\ -0.6 & -0.8 \end{bmatrix} \quad \mathbf{E} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

Fill in the blanks in the sentences below. We are told that there is a solution in each case.

Matrix _____ represents a scaling.

Matrix _____ represents a projection.

Matrix _____ represents a shear.

Matrix _____ represents a reflection.

Matrix _____ represents a rotation.

28. Each of the linear transformations in parts (a) through (e) corresponds to one (and only one) of the matrices **A** through **J**. Match them up.

    a. Scaling   b. Shear   c. Rotation   d. Projection   e. Reflection

$$\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} -0.6 & 0.8 \\ -0.8 & -0.6 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} \quad \mathbf{E} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{bmatrix} \quad \mathbf{G} = \begin{bmatrix} 0.6 & 0.6 \\ 0.8 & 0.8 \end{bmatrix} \quad \mathbf{H} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \quad \mathbf{I} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad \mathbf{J} = \begin{bmatrix} 0.8 & -0.6 \\ 0.6 & -0.8 \end{bmatrix}$$

Section 2.4:

40. Show that if a square matrix **A** has two equal columns, then **A** is not invertible.

41. Which of the following linear transformations T from $\mathbf{R}^3$ to $\mathbf{R}^3$ are invertible? Find the inverse if it exists.
   a. Reflection about a plane.
   b. Projection onto a plane.
   c. Scaling by a factor of 5 [i.e., $T(\mathbf{v}) = 5\mathbf{v}$, for all vectors **v**].
   d. Rotation about an axis.

42. A square matrix is called a permutation matrix if it contains a 1 exactly once in each row and in each column, with all other entries being 0. Examples are the identity matrix $\mathbf{I}_n$ and $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Are permutation matrices invertible? If so, is the inverse a permutation matrix as well?

43. Consider two invertible $n \times n$ matrices **A** and **B**. Is the linear transformation $\mathbf{y} = \mathbf{A}(\mathbf{Bx})$ invertible? If so, what is the inverse? [Hint: Solve the equation $\mathbf{y} = \mathbf{A}(\mathbf{Bx})$ first for **Bx** and then for **x**.]

53. Let $\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 3 & 5 \end{bmatrix}$ in all parts of this problem.

   a. Find a scalar λ (lambda) such that the matrix $\mathbf{A} - \lambda \mathbf{I}_2$ fails to be invertible. There are two solutions; choose one and use it in parts (b) and (c).
   b. For the λ you chose in part (a), find the matrix $\mathbf{A} - \lambda \mathbf{I}_2$; then find a nonzero vector **x** such that $(\mathbf{A} - \lambda \mathbf{I}_2)\mathbf{x} = \mathbf{0}$. (This can be done, since $\mathbf{A} - \lambda \mathbf{I}_2$ fails to be invertible.)
   c. Note that the equation $(\mathbf{A} - \lambda \mathbf{I}_2)\mathbf{x} = \mathbf{0}$ can be written as $\mathbf{Ax} - \lambda \mathbf{x} = \mathbf{0}$, or, $\mathbf{Ax} = \lambda \mathbf{x}$. Check that the equation $\mathbf{Ax} = \lambda \mathbf{x}$ holds for your λ from part (a) and your **x** from part (b).

Extra problems for those interested in economics (<u>not to be turned in</u>):
Section 2.4:

49. ***Input-Output Analysis***. (This exercise builds on Exercises 1.1.20, 1.2.37, 1.2.38, and 1.2.39.) Consider the industries J_1, J_2, ..., J_n in an economy. Suppose the consumer demand vector is **b**, the output vector is **x** and the demand of the jth industry is $\mathbf{v}_j$. (The ith component a_{ij} of $\mathbf{v}_j$ is the demand industry J_j puts on industry J_i, per unit of output of J_j.) As we have seen in Exercise 1.2.38, the output **x** just meets the aggregate demand if

$$\underbrace{x_1\mathbf{v}_1 + x_2\mathbf{v}_2 \cdots x_n\mathbf{v}_n + \mathbf{b}}_{\text{aggregate demand}} = \underset{\text{output}}{\mathbf{x}} .$$

This equation can be written more succinctly as

$$\begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ | & | & & | \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \mathbf{b} = \mathbf{x}$$

or $\mathbf{A}\mathbf{x} + \mathbf{b} = \mathbf{x}$. The matrix **A** is called the *technology matrix* of this economy; its coefficients a_{ij} describe the inter-industry demand, which depend on the technology used in the production process. The equation

$$\mathbf{A}\mathbf{x} + \mathbf{b} = \mathbf{x}$$

describes a linear system, which we can write in the customary form:

$$\mathbf{x} - \mathbf{A}\mathbf{x} = \mathbf{b} \quad \Rightarrow \quad \mathbf{I}_n\mathbf{x} - \mathbf{A}\mathbf{x} = \mathbf{b} \quad \Rightarrow \quad (\mathbf{I}_n - \mathbf{A})\mathbf{x} = \mathbf{b}$$

If we want to know the output **x** required to satisfy a given consumer demand **b** (this was our objective in the previous exercises), we can solve this linear system, preferably via the augmented matrix.

   In economics, however, we often ask the other questions: If **b** changes, how will **x** change in response. If the consumer demand on one industry increases by 1 unit and the consumer demand on the other industries remains unchanged, how will **x** change? If we ask questions like these, we think of the output **x** as a *function* of the consumer demand **b**.

   If the matrix $(\mathbf{I}_n - \mathbf{A})$ is invertible, we can express **x** as a function **b** (in fact, as a linear transformation):

$$\mathbf{x} = (\mathbf{I}_n - \mathbf{A})^{-1}\mathbf{b}$$

   a. Consider the economy of Israel in 1958 (discussed in Exercise 1.2.39). Find the technology matrix **A**, the matrix $(\mathbf{I}_n - \mathbf{A})$, and its inverse $(\mathbf{I}_n - \mathbf{A})^{-1}$.

   b. In the example discussed in part (a), suppose the consumer demand on agriculture (Industry 1) is 1 unit (1 million pounds), and the demands on the other two industries are zero. What output **x** is required in this case? How does your answer relate to the matrix $(\mathbf{I}_n - \mathbf{A})^{-1}$?

   c. Explain, in terms of economics, why the diagonal elements of the matrix $(\mathbf{I}_n - \mathbf{A})^{-1}$ you found in part (a) must be at least 1.

   d. If the consumer demand on manufacturing increases by 1 (from whatever it was), and the consumer demand on the other two industries remains the same, how will the output have to change? How does your answer relate to the matrix $(\mathbf{I}_n - \mathbf{A})^{-1}$?

   e. Using your answers in parts (a) through (d) as a guide, explain in general (not just for this example) what the columns and the entries of the matrix $(\mathbf{I}_n - \mathbf{A})^{-1}$ tell you, in terms of economics. Those who have

   studied multivariable calculus may wish to consider the partial derivatives $\dfrac{\partial x_i}{\partial b_j}$.

50. This exercise refers to Exercise 49a. Consider the entry $k = a_{11} = 0.293$ of the technology matrix **A**. Verify that the entry in the first row and the first column of $(\mathbf{I}_n - \mathbf{A})^{-1}$ is the value of the geometric series $1 + k + k^2 + \cdots$. Interpret this observation in terms of economics.