Math E-21b – Spring 2025 – Homework #14

[You should do all of these problems, but they are not to be turned in.]

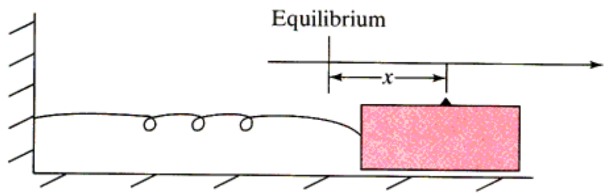
Problem 1. (a) Find all complex solutions of the system $\dfrac{d\mathbf{x}}{dt}=\begin{bmatrix}3 & -2\\ 5 & -3\end{bmatrix}\mathbf{x}$ in the form $\mathbf{x}(t)=c_1e^{\lambda_1 t}\mathbf{w}_1+c_2e^{\lambda_2 t}\mathbf{w}_2$ with complex eigenvalues λ_1 and λ_2 and corresponding complex eigenvectors $\mathbf{w}_1$ and $\mathbf{w}_2$.

(b) Now express all solutions in the real form $\mathbf{x}(t)=e^{at}[c_1\cos(bt)\mathbf{v}_1+c_2\sin(bt)\mathbf{v}_2]$ for appropriate (real) vectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

Problem 2. Determine the stability of the system $\dfrac{d\mathbf{x}}{dt}=\begin{bmatrix}-1 & 2\\ 3 & -4\end{bmatrix}\mathbf{x}$.

Problem 3. Determine the stability of the system $\dfrac{d\mathbf{x}}{dt}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -1 & -1 & -2\end{bmatrix}\mathbf{x}$.

Problem 4. Consider the following mass-spring system:



Let $x(t)$ be the deviation of the block from the equilibrium position at time t. Consider the velocity $v(t)=\dfrac{dx}{dt}$ of the block. There are two forces acting on the mass: The spring force F_s, which is assumed to be proportional to the displacement x, and the force F_f of friction, which is assumed to be proportional to the velocity

$$F_s=-px\ ,\quad F_f==-qv$$

where $p>0$ and $q\geq 0$. (q is 0 if the oscillation is frictionless.) The total force acting on the mass is thus

$$F=F_s+F_f=-px-qv\,.$$

By Newton's second law of motion, we have

$$F=ma=m\frac{dv}{dt}\,,$$

where a represents acceleration and m the mass of the block. Combining the last two equations, we find that

$$m\frac{dv}{dt}=-px-qv \quad\text{or}\quad \frac{dv}{dt}=-\frac{p}{m}x-\frac{q}{m}v\,.$$

Let $b=\dfrac{p}{m}$ and $c=\dfrac{q}{m}$ for simplicity. Then the dynamics of this mass-spring system are described by the system

$$\left\{\begin{aligned}\frac{dx}{dt}&=v\\ \frac{dv}{dt}&=-bx-cv\end{aligned}\right\}\quad (b>0,\ \ c\geq 0).$$

Sketch a phase portrait for this system in each of the following cases, and describe briefly the significance of your trajectories in terms of the movement of the block. Comment on the stability in each case.

(a) $c=0$ (frictionless). Find the period. (b) $c^2<4b$ (underdamped). (c) $c^2>4b$ (overdamped).

Exercises from Supplement on Nonlinear Systems and Linearization

Problem 5. The interaction of two species of animals is modeled by $\left\{ \begin{aligned} \frac{dx}{dt} &= x(2-x+y) \\ \frac{dy}{dt} &= y(4-x-y) \end{aligned} \right\}$ for $x \geq 0$ and $y \geq 0$.

a) Sketch a phase portrait for this system. Make sure that your sketch clearly shows the nullclines and the equilibria.

b) There is one equilibrium point (a,b) with $a>0$ and $b>0$. Find the Jacobian matrix **J** of the system at that point.

c) Determine the stability of the equilibrium point (a,b) discussed in part (b).

Problem 6. Consider the system $\left\{ \begin{aligned} \frac{dx}{dt} &= x^2+y^2-1 \\ \frac{dy}{dt} &= xy \end{aligned} \right\}$.

Sketch a phase plane for this system. Make sure that your sketch clearly shows the nullclines and the equilibria. Which equilibria are stable (using linearization and Jacobian analysis)?